

WORKING PAPER

Optimal Portfolio and Spending Rules for Endowment Funds

M. Kashif F. Menoncin I. Owadally (3)

WPDEM 3/2018

Via S. Faustino 74/b, 25122 Brescia - Italy Tel +39 030 2988742 email: paqricerca.dem@unibs.it

Optimal portfolio and spending rules for endowment funds

Muhammad Kashif⁽¹⁾ Francesco Menoncin⁽²⁾ Iqbal Owadally⁽³⁾

(1) Department of Economics and Management, University of Brescia, Italy, muhammad.kashif@unibg.it

(2) Department of Economics and Management, University of Brescia, Italy, francesco.menoncin@unibs.it

(3) Cass Business School, City University London, United Kingdom, iqbal@city.ac.uk

Abstract

We investigate the role of different spending rules in a dynamic asset allocation model for an endowment fund. In particular, we derive the optimal portfolios under the consumption-wealth ratio rule (CW strategy) and the hybrid rule (hybrid strategy) and compare them with a theoretically optimal (Merton's) strategy for both spending and portfolio allocation. Furthermore, we show that the optimal portfolio is less risky with habit as compared with the optimal portfolio without habit. Similarly, the optimal portfolio under hybrid strategy is less risky than both CW and Merton's strategy for given set of constant parameters. Thus, endowments following hybrid spending rule use asset allocation to protect spending. Our calibrated numerical analysis on US data shows that the consumption under hybrid strategy is less volatile as compared to other strategies. However, hybrid strategy comparatively outperforms the conventional Merton's strategy and CW strategy when the market is highly volatile but under-performs them when there is a low volatility. Overall, the hybrid strategy is effective in terms of stability of spending and intergenerational equity because, even if it allows fluctuation in spending in the short run, it guarantees the convergence of spending towards its long term mean.

Keywords: Spending rules, Endowments, Asset allocation, Habit formation, Bellman's equation.

1 Introduction

University endowments rank among the largest institutional investors. In 2016, the National Association of College and University Business Officers (NACUBO) estimated that its member organizations held \$515 billion in endowment assets (NACUBO [2016]). Universities and colleges set up endowment funds to ensure a reasonably smooth earnings and consequent smooth stream of spending for current and future beneficiaries to preserve equity among generations (also called intergenerational equity, see Tobin [1974]). To achieve this objective, many endowment funds set pre-defined spending rules. Hansmann [1990] addresses the reason why endowments would need such rules and concludes that endowments are established to fulfill some basic

purposes which include: (i) ensure the support of the parent institution in its ongoing mission, (ii) protect its reputation and intellectual freedom, and (iii) hedge against financial shocks. In general, following a predefined spending rule is not optimal. As a result, endowment funds may invest too much in riskless assets which generally have lower returns.

Kaufman and Woglom [2005] discuss the role of spending rules in endowment funds. They analyze the spending rules based on the inflation method, banded inflation, and hybrid method, using Monte Carlo simulations in a scenario of volatile and uncertain asset returns. Ennis and Williamson [1976] present different spending rules adopted by endowment funds along with their historical spending patterns. Sedlacek and Jarvis [2010] provide an analysis of endowment's current practices and spending policies with their relative merit and demerits.

In this paper, we examine investment strategies under two spending rules in particular: (i) the consumption-wealth ratio rule, a simplified form of moving average method, and (ii) the weighted average or hybrid rule which is more commonly used by large endowment funds like Yale and Stanford (as stated in Cejnek et al. [2014]). Under consumption-wealth ratio (CW) rule, the spending is a percentage of the market value of the fund while hybrid rule calculates the spending as a weighted average of the inflation method and the moving average method. To analyze the effect of the spending rules on risk taking, we derive the optimal portfolio under the above mentioned spending rules for hyperbolic absolute risk aversion utility function and compare them with the classical Merton's optimal investment and consumption.

An endowment fund may exhibit intertemporal preferences where past spending generates a desire (or need) to maintain the same level. Such a behavior can be modeled through habit formation or habit persistence, which is a more general case of the subsistence level consumption. Actually, the habit can be interpreted as a time varying subsistence consumption.

In Merton's case, we consider a general form of utility function including both the case of habit formation and subsistence level. Under the spending rules mechanism, endowments following hybrid spending rule protect spending by investing in a less risky portfolio than Merton's. Similarly, investment in the risky asset with habit is less than the investment without habit. Thus, their strategy is similar to proportional portfolio insurance where the fund invests in safe assets to maintain the value needed for having smooth payouts over time.

We calibrate parameters over three different time horizons to investigate the effectiveness of the spending rules. The hybrid strategy comparatively outperforms the conventional Merton's and CW strategies when

the market is highly volatile but under-performs it when there is a low volatility.

The dynamic asset allocation and consumption studied in this paper is built upon the seminal work by Samuelson [1969] and Merton [1969], who present an optimal strategy for a market with constant investment opportunities with additive time-separable utility function. This preliminary work was later extended by Merton [1971], to a more general utility function which includes the income generated by non-capital gains sources.

This paper is related to a number of other works in the literature. Merton [1993] applies continuous time framework to the endowment fund's problem and derives optimal expenditures and asset allocation strategy that include non-endowed funds as a part of the total university's wealth. He concludes that endowment funds prefer a safer portfolio in the presence of non-financial income risk. Dybvig [1999] suggests that an endowment fund spending can be sustainable if some TIPS (Treasury Inflation Protection Securities) are hold in portfolio. Bajeux-Besnainou and Ogunc [2006] address the asset allocation problem of an endowment fund by including, in the objective function, a minimum spending amount up-rated with inflation and obtain an explicit formula for optimal spending and portfolio allocation rules. Constantinides [1990] applies the habit formation to the equity puzzle problem and shows that the high equity premium with low risk aversion can be explained by the presence of habit formation. Gong and Li [2006] considers the general optimal investment/consumption problem for an agent with constrained consumption due to habit formation or precommitment. They found that investment in nominal bonds and equity is initially proportional to the excess wealth over the lower bound imposed due to subsistence level of consumption and then it increases nonlinearly with wealth. Munk [2008] studies the optimal strategies with general asset price dynamics under two special cases of time varying investment opportunities: stochastic interest rate, and mean-reverting stock returns. He shows that, in order to finance the habit, investment in bonds and cash is more effective than in stocks.

The rest of the paper is structured in the following way. Section 2 describes the various spending strategies commonly applied by endowment funds, while Section 3 introduces the general framework for market dynamics, preferences of endowment fund, and endowment fund investment strategies. Section 4 focuses on the results of optimal investment and spending under different strategies. Section 5 presents a numerical application of the previous results, and, finally, Section 6 concludes. Some technical derivations are left to Appendix A.

Categories	Method	Description	
1. Simple Rules	Income-based	Spend the whole current income.	
	Consumption- wealth ratio	Spend either the predetermined percentage of the market value of the fund or decide the percentage every year.	
2. Inflation-Based Rules	Inflation- protected	Spending grows at the rate of inflation.	
	Banded- Inflation	Same as inflation-protected but with the upper and lower bands.	
3. Smoothing Rules	Moving Average	Pre-specified percentage of moving average of market values, generally based on three-years starting market values.	
	Spending Reserve	5-10 percent of the market value is held in the reserve account and then invested in 90-day Treasury bills. The amount is withdrawn only when the fund's performance is below target.	
	Stabilization Fund	The excess endowment returns are used to make a fund which is then used to control the long term growth of the total endowment.	
4. Hybrid Rules	Weighted Average or Hybrid (Yale/Stanford) Method	Spending is calculated as the weighted average of spending adjusted for inflation and spending under the moving average method.	

Table I: Endowment Spending Rules. Source: Sedlacek and Jarvis [2010]

2 Spending rules

In practice, there are various spending rules actually followed by endowment funds. According to survey data Cejnek et al. [2014] based on the original work by Sedlacek and Jarvis [2010], the spending rules are divided into four categories: (i) simple rules, (ii) inflation-based rules, (iii) smoothing rules, and (iv) hybrid rules. These rules are summarized in Tab. I.

In the following subsections we consider the following rules in detail:

- Inflation rule;
- Moving average method;

• Weighted average or hybrid method.

2.1 Inflation rule

Inflation rule is devised to acknowledge the corrosive effects of inflation and it aims to protect the purchasing power of endowment fund. The objective of endowment fund is not the mere preservation of the fund but to strive for a value addition. It can be achieved as long as the overall return from the portfolio exceeds the rate of inflation. Inflation rule increases the previous period's spending at the predetermined inflation rate. Spending in a year equals the spending in the previous year, increased at the inflation rate λ :

$$c(t)=c_0e^{\lambda t},$$

where the initial value of consumption is a fixed ratio of wealth, i.e. $c_0 = yR_0$. The differential of c(t) is

$$dc(t) = \lambda c(t)dt. \tag{1}$$

The inflation method is static and trivial, so we will not consider it separately here, however it is included in the hybrid method along with the moving average method.

2.2 Moving average method

The most popular and commonly used spending rule is the moving average. Dimmock [2012] reports that typically this rule is based on pre-specified percentage of moving average of 3-years quarterly market value. The most desirable property of this rule is that it saves some income and reinvests it. This method is used to smooth the volatility in spending. However, the method is flawed because it uses the market value of the endowment. Therefore, when the endowment value is rising, the institution may spend more than it is prudent and when endowment values are falling sharply, this method calls for a budget cut that may curtail the institution's mission.

In discrete-time, this spending rule can be algebraically written as follows

$$c(t) = \frac{y}{q} (R(t) + \dots + R(t - (q - 1)))$$

In continuous-time, instead, we can write

$$c(t) = \frac{y}{q} \int_{t-q}^{t} R(\tau) d\tau.$$

Here, for the sake of simplicity, we take the limit of the previous rule for q which tends towards zero:¹

$$c(t) = \lim_{q \to 0} \frac{y}{q} \int_{t-q}^{t} R(\tau) d\tau = \lim_{q \to 0} yR(t-q) = yR(t),$$
(2)

which is the same as the consumption-wealth ratio rule stated in Tab. I.

The goal of moving average rule is to dampen the volatility of spending. Consequently, during the period of boom, this process results in an accelerating curve of upward spending and it causes a false sense of security of sustainable spending. The smoothing effect of this rule is limited and it may give a misplaced belief that the higher spending levels are sustainable. Furthermore, the resulting shrinkage in endowment values due to market decline results into the deep cuts in spending.

2.3 Weighted average or hybrid method

The weighted average method is generally followed by the large endowment funds and it is also known as Yale/Stanford rule. It is a weighted average of the inflation method and the moving average method.

In discrete-time it is given by

$$c(t+1) = \omega c(t) e^{\lambda} + (1-\omega) \frac{y}{q} \left(R(t+1) + \dots + R(t+1-(q-1)) \right),$$
(3)

where ω is the weight. We can simplify it by considering q = 1 to get

$$c(t+1) = \omega c(t) e^{\lambda} + (1-\omega) y R(t+1).$$
(4)

If we ignore inflation, i.e. $\lambda = 0$, we obtain

$$c(t+1) = \omega c(t) + (1-\omega) yR(t+1).$$
(5)

We make the above process stationary by assuming $|(1 - \omega)y| < 1$.

¹De l'Hôpital theorem is used, by recalling that $\frac{\partial}{\partial q} \int_{t-q}^{t} R(\tau) d\tau = R(t-q).$

Remark 1. If we take the limit $t + 1 \rightarrow t$, we get the same consumption-wealth ratio as in (2)

$$c(t) = yR(t).$$

Under the hybrid rule, during the boom, the spending will not increase as fast as compared with the moving average rule. Conversely, this rule does not call for spending cuts as deep as the moving average method. Evidence suggests that more and more institutions are changing their spending rules to inflation-based and hybrid method from moving average method Sedlacek and Jarvis [2010].

We will only consider two spending rules, the consumption-wealth ratio rule, and the hybrid spending rule given by (2) and (5), respectively for our optimal investment and spending strategies.

3 General framework

Endowment funds usually invest in a variety of assets, however for the purpose of tractability we examine the aforementioned spending rules in the simplest framework. We consider two types of assets, a riskless asset and a risky asset in a complete and arbitrage free, continuously open financial market. On the financial market two assets are listed:

• a riskless asset G(t) which evolves according to

$$\frac{dG(t)}{G(t)} = r(t)dt,$$
(6)

where $G(t_0) = 1$, and which is the *numéraire* and r(t) is the instantaneous nominal interest rate;

• a risky asset S(t) having the price dynamics given by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t).$$
⁽⁷⁾

The endowment fund holds $\theta_S(t)$ units of the risky asset S(t), and $\theta_G(t)$ units of risk-less assets G(t). Thus, at any instant in time *t*, the investor's wealth R(t) is given by

$$R(t) = \theta_G(t)G(t) + \theta_S(t)S(t).$$

The differential of wealth can be written as

$$dR(t) = (R(t)r(t) + \theta_{S}(t)S(t)(\mu - r(t)) - c(t))dt + \theta_{S}(t)S(t)\sigma dW(t),$$
(8)

where c(t) is the consumption or spending, following one of the rules mentioned in Section 2 except when it is also a decision variable.

3.1 Endowment fund investment strategies

We consider the following three strategies:

- Merton's strategy: Both investment and spending are decision variables.
- Consumption-wealth ratio (CW) strategy: Investment is the only decision variable and spending is given by the fixed consumption-wealth ratio rule.
- **Hybrid strategy:** Investment is the only decision variable and spending is given by weighted average or hybrid method.

3.2 General settings

According to Fraser and Jennings [2010], an endowment fund must define its investment policy statement identifying the investment beliefs, specific investment objectives, re-balancing policy and performance benchmark which are evaluated periodically. Since an endowment fund must report its performance for each accounting period, it is reasonable to consider the optimization problem for a finite time horizon [t, T]. If we assume the objective of an endowment fund is to maximize the sum of expected utility of spending c(t) and the expected utility of the final wealth R(T) then it can be stated as

$$\max_{\pi(t)} \mathbb{E}_{t} \left[\int_{t}^{T} U_{c}(c(t)) e^{-\rho(s-t)} ds + U_{R}(R(T)) e^{-\rho(T-t)} \right],$$
(9)

where endowment fund chooses the decision variables $\pi(t)$, which may include consumption and investment depending on the strategy considered and ρ is a constant subjective discount rate. We assume the fund's preferences are defined by the following utility function

$$U(x(t)) = \frac{\left(x(t) - \alpha_X\right)^{1-\delta}}{1-\delta},\tag{10}$$

where $\delta > 1$. The Arrow-Pratt Relative Risk Aversion (ARA) index is

$$-\frac{\frac{\partial^2 U(x(t))}{\partial x(t)^2}}{\frac{\partial U(x(t))}{\partial x(t)}} = \frac{\delta}{x(t) - \alpha_X}$$

If α_X is a positive constant, then this form of the utility function belongs to Hyperbolic Absolute Risk Aversion (HARA). If $\alpha_X = 0$, ARA index is given by

$$-\frac{\frac{\partial^2 U(x(t))}{\partial x(t)^2}}{\frac{\partial U(x(t))}{\partial x(t)}} = \frac{\delta}{x(t)},$$

which belongs to Constant Relative Risk Aversion (CRRA).

If $\alpha_X \neq 0$, the utility functions belong to HARA and can be written as

$$U_{c}(c(t),h(t)) = \frac{(c(t)-h(t))^{1-\delta}}{1-\delta}, \quad U_{R}(R(T)) = \frac{(R(T)-R_{m})^{1-\delta}}{1-\delta},$$
(11)

where c(t) is the instantaneous outflow or spending from the fund, the constant R_m can be interpreted as the minimum subsistence level of wealth, whereas h(t) depends on the context, it is either a function representing the habit formation or a constant representing the subsistence level. Given the utility function (11), it is always optimal to have outflows higher than the threshold h(t), in fact when c(t) = h(t), the marginal utility of the outflow tends towards infinity and, accordingly, it is sufficient to increase it by an infinitesimal amount in order to have an infinite increase in the utility level.

The corresponding ARA indices of (11) are given by

$$-\frac{\frac{\partial^2 U_c(c(t),h(t))}{\partial c(t)^2}}{\frac{\partial U_c(c(t),h(t))}{\partial c(t)}} = \frac{\delta}{c(t) - h(t)}, \quad -\frac{\frac{\partial^2 U_R(R(T))}{\partial R(t)^2}}{\frac{\partial U_R(R(T))}{\partial R(t)}} = \frac{\delta}{R(T) - R_m},$$

respectively, which implies that the higher δ the higher the risk aversion. Moreover, the higher h(t) the higher is the risk aversion. This result shows that having a higher level of minimum outflows means that it is necessary to invest bigger amounts of wealth in the riskless asset in order to guarantee the outflows, which implies a higher risk aversion.

4 The optimal solutions

4.1 Merton's strategy

Investment and spending are both decision variables, i.e. $\pi(t) \equiv \{c(t), \theta_S(t)\}$. We can define the value function using (9) as

$$J(t,R(t)) \equiv \max_{c(t),\theta_{S}(t)} \mathbb{E}_{t} \left[\int_{t}^{T} \phi_{c} \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_{R} \frac{(R(T)-R_{m})^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right], \quad (12)$$

where ϕ_c and ϕ_R are constants which measures the subjective relevance of the utility obtained from intertemporal consumption and final wealth, respectively and h(t) is given by

$$h(t) = \begin{cases} h_0 e^{-\int_0^t \beta(u)du} + \int_0^t \alpha(s)c(s) e^{-\int_s^t \beta(u)du}ds, & \text{habit formation,} \\ h, & \text{subsistence level,} \\ 0, & \text{classical problem,} \end{cases}$$
(13)

where h_0 is the initial minimum amount of outflow, $\alpha(t)$ is the weighting function providing the relative importance to the past outflow in computing the threshold h(t), while $\beta(t)$ is a discount rate. In habit formation case, h(t) can be rewritten in continuous time as

$$dh(t) = (\alpha(t)c(t) - \beta(t)h(t))dt.$$
(14)

Proposition 1. *Given the state variable wealth* R(t) *described in (8), the optimal consumption and portfolio solving problem (12) are*

• in the case with habit formation:

$$c(t)^{*} = h(t) + \phi_{c}^{\frac{1}{\delta}} \frac{(R(t) - h(t)B(t))(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t)},$$
(15)

$$\theta_S(t)^* = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - h(t)B(t)}{\delta},\tag{16}$$

where

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta - 1}{\delta}r(u) + \frac{\rho}{\delta} + \frac{(\delta - 1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} + \int_t^T \phi_c^{\frac{1}{\delta}} \left(1 + B(s)\alpha(s)\right)^{\frac{\delta - 1}{\delta}} e^{-\int_t^s \left(\frac{\delta - 1}{\delta}r(u) + \frac{\rho}{\delta} + \frac{(\delta - 1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} ds B(t) = R_m e^{-\int_t^T (-\alpha(u) + \beta(u) + r(u)) du} + \int_t^T e^{-\int_t^s (-\alpha(u) + \beta(u) + r(u)) du} ds.$$

• in the case of a subsistence level:

$$c(t)^{*} = h + \phi_{c}^{\frac{1}{\delta}} \frac{R(t) - hB(t)}{A(t)}, \quad \theta_{S}(t)^{*} = \frac{\mu - r(t)}{S(t)\sigma^{2}} \frac{R(t) - hB(t)}{\delta}.$$
(17)

• in the classical case:

$$c(t)^{*} = \phi_{c}^{\frac{1}{\delta}} \frac{R(t)}{A(t)}, \quad \theta_{S}(t)^{*} = \frac{\mu - r(t)}{S(t)\sigma^{2}} \frac{R(t)}{\delta},$$
 (18)

where for both the subsistence level case and the classical case we have

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta}r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} + \phi_c^{\frac{1}{\delta}} \int_t^T e^{-\int_t^s \left(\frac{\delta-1}{\delta}r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} ds,$$

$$B(t) = R_m e^{-\int_t^T r(u) du} + \int_t^T e^{-\int_t^s r(u) du} ds.$$

Proof. See Appendix A.1.

The function A(t) in the optimal solutions is the weighted sum of two discount factors: (i) the discount factor for the final date T multiplied by $\phi_R^{\frac{1}{\delta}}$, and (ii) a kind of intertemporal discount factor for the intertemporal utility, multiplied by $\phi_c^{\frac{1}{\delta}}$. The function B(t) in the optimal solutions, is the sum of two terms, the subsistence wealth R_m appropriately discounted from time T and a sum of discount factors. We can see that habit formation has an effect on the optimal portfolio of the risky asset, as it changes the allocation due to the reason that the riskless asset (treasury) is comparatively a better investment than the risky asset (stock) to ensure that the future consumption will not decline below the habit level.

We consider the optimal investment and consumption in the habit formation case defined in (16) and (15), respectively, in detail below:

Assumption 1. We assume that all the parameters α , β , μ , σ , and r are constant over time and additionally $r - \alpha + \beta > 0$.

Corollary 1. Under Assumption 1, (i) The wealth process is itself a Markov process and the functions A(t) and B(t) can be written as

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta}r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2 \sigma^2}\right)(T-t)} + \int_t^T \phi_c^{\frac{1}{\delta}} (1+B(s)\alpha)^{\frac{\delta-1}{\delta}} e^{-\left(\frac{\delta-1}{\delta}r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2 \sigma^2}\right)(s-t)} ds,$$

$$B(t) = R_m e^{-(r-\alpha+\beta)(T-t)} + \int_t^T e^{-(r-\alpha+\beta)(s-t)} ds = R_m e^{-(r-\alpha+\beta)(T-t)} + \frac{1-e^{-(r-\alpha+\beta)(T-t)}}{r-\alpha+\beta},$$
(19)

and finally, if we substitute the value of B(t) into A(t), we get

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta}r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2 \sigma^2}\right)(T-t)} + \int_t^T \phi_c^{\frac{1}{\delta}} (1 + R_m e^{-(r-\alpha+\beta)(T-s)} + \frac{1 - e^{-(r-\alpha+\beta)(T-s)}}{r-\alpha+\beta} \alpha)^{\frac{\delta-1}{\delta}} e^{-\left(\frac{\delta-1}{\delta}r + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu-r)^2}{2\delta^2 \sigma^2}\right)(s-t)} ds.$$
(20)

(ii) The optimal consumption and portfolio are

$$c(t)^{*} = h(t) + \phi_{c}^{\frac{1}{\delta}} \frac{(R(t) - h(t)B(t))(1 + B(t)\alpha)^{-\frac{1}{\delta}}}{A(t)},$$
(21)

$$\theta_{S}(t)^{*} = \frac{\mu - r}{S(t)\sigma^{2}} \frac{R(t) - h(t)B(t)}{\delta},$$
(22)

where A(t) and B(t) are given by (19) and (20).

The term

$$\frac{1-e^{-(r-\alpha+\beta)(T-s)}}{r-\alpha+\beta}$$

is positive and decreasing over time as $r - \alpha + \beta > 0$.

(iii) The optimal portfolio is less risky with habit as compared with the optimal portfolio without habit:

$$S(t)\theta_S(t)^* < S(t)\theta_S(t)^*|_{h(t)=0}.$$

Proof. The optimal portfolio $S(t)\theta_S(t)^*$ is given by

$$\frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - h(t)B(t)}{\delta},$$

Since by construction $h(t) \ge 0$ and $B(t) \ge 0$, thus

$$\frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - h(t)B(t)}{\delta} < \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t)}{\delta}.$$

Assumption 2. We assume $T \rightarrow \infty$.

Corollary 2. Under Assumptions 1 and 2, the optimal consumption and portfolio are

$$c(t)^* = h(t) + \phi_c^{\frac{1}{\delta}} \frac{(R(t) - h(t)\frac{1}{r - \alpha + \beta})(1 + \frac{\alpha}{r - \alpha + \beta})^{-\frac{1}{\delta}}}{A(t)},$$

$$\theta_{S}(t)^{*} = \left(R(t) - \frac{h(t)}{r - \alpha + \beta}\right) \frac{\mu - r}{S(t)\sigma^{2}\delta}.$$

In this case the dynamics of optimal wealth and habit are

$$dR(t) = \left(R(t)r(t) + \left(R(t) - \frac{h(t)}{r - \alpha + \beta} \right) \frac{(\mu - r(t))^2}{\sigma^2 \delta} - h(t) \right)$$
$$+ \phi_c^{\frac{1}{\delta}} \frac{\left(R(t) - h(t) \frac{1}{r - \alpha + \beta} \right) \left(1 + \frac{\alpha}{r - \alpha + \beta} \right)^{-\frac{1}{\delta}}}{A(t)} \right) dt$$
$$+ \left(R(t) - \frac{h(t)}{r - \alpha + \beta} \right) \frac{\mu - r(t)}{\sigma \delta} dW(t),$$

$$dh(t) = (\beta - \alpha) \left(\frac{\alpha}{\beta - \alpha} \phi_c^{\frac{1}{\delta}} \frac{\left(R(t) - h(t) \frac{1}{r - \alpha + \beta} \right)^{\left(1 + \frac{\alpha}{r - \alpha + \beta}\right)^{-\frac{1}{\delta}}}}{A(t)} - h(t) \right) dt,$$

where we can see that the habit h(t) is a mean reverting process if $\beta - \alpha > 0$ and is instead exploding if $\beta - \alpha < 0$.

4.2 CW strategy

In this case, the investment $\theta_S(t)$ is the only decision variable and the spending c(t) is given by

$$c(t) = yR(t), \tag{23}$$

therefore we can put $\phi_c = 0$ and $\phi_R = 1$ in (12), then the value function can be defined as

$$J(t,R(t)) \equiv \max_{\theta_{\mathcal{S}}(t)} \mathbb{E}_t \left[\frac{(R(T) - R_m)^{1 - \delta}}{1 - \delta} e^{-\rho(T - t)} \right].$$
(24)

Proposition 2. Given the state variable R(t) and c(t) described in (8) and (23), respectively, the optimal portfolio solving problem (24) is

$$\theta_S(t)^* = \frac{\mu - r(t)}{S(t)\sigma^2} \frac{R(t) - B(t)}{\delta},\tag{25}$$

where

$$\begin{split} A(t) &= e^{-\int_t^T (\frac{\delta-1}{\delta}r(s) - \frac{\delta-1}{\delta}y + \frac{\delta-1}{2\delta^2} \frac{(\mu-r(s))^2}{\sigma^2} - \frac{\rho}{\delta})ds},\\ B(t) &= R_m e^{-\int_t^T (r(s) - y)ds}, \end{split}$$

and y is the constant defined in (23).

Proof. See Appendix A.2.

The function A(t) in proposition 2 is a discount factor calculated as a sum of the accumulated interest rate and Sharpe ratio diminished by the constant y. The function B(t) in the optimal solutions, is the subsistence wealth R_m appropriately discounted by a discount factor.

Corollary 3. Under Assumptions 1,2 and r > y, the optimal portfolio (25) becomes

$$\theta_S(t)^* = rac{\mu - r}{S(t)\sigma^2} rac{R(t)}{\delta}.$$

In this case the dynamics of optimal wealth is

$$\frac{dR(t)}{R(t)} = \left(r(t) + \frac{1}{\delta}\frac{(\mu - r)^2}{\sigma^2} - y\right)dt + \frac{1}{\delta}\frac{\mu - r}{\sigma}dW(t).$$

4.3 Hybrid strategy

In this strategy, investment is the only decision variable $\pi \equiv \theta_S(t)$, while spending evolves according to the weighted average spending rule (5). We can write the wealth dynamics (8) in a discrete time as

$$R(t+1) = R(t)(1+r(t)) + \theta_{S}(t)S(t)(\mu - r(t)) - c(t) + \theta_{S}(t)S(t)\sigma Z(t),$$
(26)

and the consumption in a discrete time is given by

$$c(t+1) = \omega c(t) + (1-\omega) y R(t+1).$$
(27)

We substitute (26) into (27) to get

$$c(t+1) = \omega c(t) + (1-\omega) y(R(t)(1+r(t)) + \theta_{S}(t)S(t)(\mu - r(t)) - c(t) + \theta_{S}(t)S(t)\sigma Z(t)),$$

which can be rewritten in continuous time as

$$dc(t) = (1 - \omega) (1 + y) \left(\frac{yR(t)(1 + r(t)) + y\theta_S(t)S(t)(\mu - r(t))}{1 + y} - c(t) \right) dt + (1 - \omega) y\theta_S(t)S(t)\sigma dW(t).$$
(28)

If we set $a \equiv (1 - \omega)y$ then we can write

$$dc(t) = (1 - \omega) (1 + y) \left(\frac{yR(t)(1 + r(t)) + y\theta_S(t)S(t)(\mu - r(t))}{1 + y} - c(t) \right) dt$$
$$+ a\theta_S(t)S(t)\sigma dW(t),$$

where we see that the consumption is a mean reverting process, whose strength of mean reversion is $(1 - \omega)(1 + y)$. The consumption reverts towards its long term mean:

$$\frac{yR(t)(1+r(t))+y\theta_{\mathcal{S}}(t)S(t)(\mu-r(t))}{1+y},$$

which depends on the portfolio choice. The higher the value of ω , the more slowly c(t) converges towards its long term mean and vice versa.

As the consumption is given by (28), we include it as an additional state variable and put $\phi_c = 0$ and $\phi_R = 1$ in (12), hence the value function can be defined as

$$J(t,R(t),c(t)) \equiv \max_{\theta_{S}(t)} \mathbb{E}_{t} \left[\frac{(R(T)-R_{m})^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$
⁽²⁹⁾

Proposition 3. Given the state variables R(t) and c(t) as in (8) and (28) and under the assumption that interest rate r is constant the optimal portfolio solving problem (29) is

$$\theta_{S}(t)^{*} = \frac{\mu - r}{S(t)\sigma^{2}} \frac{R(t) - B(t,c(t))}{\delta(1 - \eta^{*}a)},$$
(30)

where

$$B(t,c(t)) = \eta^* c(t) + R_m e^{-(r-\eta^* a(1+r))(T-t)},$$

and η^* takes one of the following values

$$\eta = \frac{(1+r+a-\omega) \pm \sqrt{(1+r+a-\omega)^2 - 4a(1+r)}}{2a(1+r)},$$

such that $-\infty < \eta^* < \frac{1}{a}$ and $a = y(1 - \omega) < 1$.

Proof. See Appendix A.3.

Corollary 4. Under Assumptions 1,2 and if $r > \eta^* a (1+r)$, the optimal portfolio (30) becomes

$$\theta_{S}(t)^{*} = \frac{\mu - r}{S(t)\sigma^{2}} \frac{R(t) - \eta^{*}c(t)}{\delta(1 - \eta^{*}a)}.$$

In this case the dynamics of optimal wealth and optimal consumption are

$$dR(t) = \left(R(t)r + \left(\frac{(\mu - r)^2}{\sigma^2}\frac{R(t) - \eta^* c(t)}{\delta(1 - \eta^* a)}\right) - c(t)\right)dt + \left(\frac{\mu - r}{\sigma}\frac{R(t) - \eta^* c(t)}{\delta(1 - \eta^* a)}\right)dW(t),$$

$$\begin{split} dc(t) &= \left(aR(t)(1+r) + a\left(\frac{(\mu-r)^2}{\sigma^2}\frac{R(t) - B(t,c(t))}{\delta(1-\eta^*a)}\right) - (1-\omega)(1+y)c(t)\right)dt \\ &+ a\left(\frac{\mu-r}{\sigma}\frac{R(t) - B(t,c(t))}{\delta(1-\eta^*a)}\right)dW(t). \end{split}$$

Corollary 5. Under Assumption 1, (i) the optimal portfolio is less risky for the Merton's strategy as compared with the optimal portfolio for CW strategy if $\alpha - \beta > y$ and r > y.²

$$\theta_{S,M}(t)^* < \theta_{S,C}(t)^*,$$

(ii) Under an additional assumption $\eta = 0$, we can write

$$\theta_{S,H}(t)^* < \theta_{S,M}(t)^* < \theta_{S,C}(t)^*.$$

Proof. We suppose the negation of the given statement is true

$$\theta_{S,M}(t)^* \geq \theta_{S,C}(t)^*.$$

As it is reasonable to assume $\mu - r > 0$, which implies $\frac{\mu - r}{S(t)\sigma^2\delta} > 0$. Therefore

$$R(t) - h(t)B(t) \ge R(t) - B(t).$$

We substitute the values of the unknown functions to obtain

$$R(t) - h(t) \left(R_m e^{-(r-\alpha+\beta)(T-t)} + \frac{1 - e^{-(r-\alpha+\beta)(T-t)}}{r-\alpha+\beta} \right) \ge R(t) - R_m e^{-(r-y)(T-t)},$$

since h(t) > 0 and $\frac{1-e^{-(r-\alpha+\beta)(T-t)}}{r-\alpha+\beta}$ is also positive and decreasing over time as we have $r - \alpha + \beta > 0$ under Assumption 1. As a consequence, the above statement can only be correct only if

$$e^{-(r-\alpha+\beta)(T-t)} < e^{-(r-y)(T-t)}$$
,

as $r - \alpha + \beta > 0$, r > y and $\alpha - \beta > y$. Therefore, we conclude that the following statement is true

²In what follows we use subscripts M,C and H with $\theta_S(t)$ or c(t) to indicate optimal portfolio or consumption under Merton's, CW and hybrid strategies, respectively.

$$\theta_{S,M}(t)^* < \theta_{S,C}(t)^*.$$

For part (ii), we suppose

$$\theta_{S,H}(t)^* \geq \theta_{S,M}(t)^*.$$

We substitute the values of the unknown functions to obtain

$$R(t) - R_m e^{-r(T-t)} \ge R(t) - h(t) R_m e^{-(r-\alpha+\beta)(T-t)} - h(t) \frac{1 - e^{-(r-\alpha+\beta)(T-t)}}{r-\alpha+\beta}.$$

We can clearly see that the left-hand side is smaller than the right-hand side, therefore we conclude that the statement to be proved is true. \Box

Corollary 6. Under Assumption 1, the optimal consumption under Merton's strategy as described in (15) can be equivalent to the consumption under CW strategy as described in (2) depending on the chosen values of constant y.

Proof. As the optimal consumption for Merton's strategy is

$$c_M(t)^* = h(t) + \phi_c^{\frac{1}{\delta}} \frac{(R(t) - h(t)B(t))(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t)},$$

where

$$B(t) = R_m e^{-\int_t^T (-\alpha(u)+\beta(u)+r(u))du} + \int_t^T e^{-\int_t^s (-\alpha(u)+\beta(u)+r(u))du}ds,$$

and

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} + \int_t^T \phi_c^{\frac{1}{\delta}} (1 + B(s)\alpha(s))^{\frac{\delta-1}{\delta}} e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{\rho}{\delta} + \frac{(\delta-1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} ds.$$

The consumption for CW strategy is given by

$$c_C(t)^* = yR(t).$$

Parameters	1997-2006	2007-2011	2012-2016
μ	0.0816	0.0117	0.1198
σ	0.1816	0.2659	0.1279
r, ρ	0.0356	0.0122	0.0011

Table II: Parameters calibrated on the S&P 500 and US 3-Month Treasury Bill time series between (i) January 2nd, 1997 and December 29th, 2006 (1997-2006), (ii) January 3rd, 2007 and December 30th, 2011 (2007-2011), and (iii) January 3rd, 2012 and December 30th, 2016 (2012-2016). Other assumptions include $R_0 = 100$, T = 10, $R_m = 0$ or 90 when $(R_m \rightarrow R_0)$ and $\delta = 2$.

Consumption under Merton's strategy and CW strategy can be equivalent if y is chosen such that

$$y = \frac{h(t)}{R(t)} + \phi_c^{\frac{1}{\delta}} \frac{(1+B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t)} - \phi_c^{\frac{1}{\delta}} \frac{h(t)B(t)(1+B(t)\alpha(t))^{-\frac{1}{\delta}}}{R(t)A(t)}.$$

if h = 0, then

$$y = \frac{\phi_c^{\frac{1}{\delta}}}{A(t)},$$

where $\phi_c^{\frac{1}{\delta}}$ can be suitably chosen to match *y*.

5 A numerical application

To illustrate the results of the preceding section, a simplified market structure is taken into account under Assumption 1 and condition that h(t) = 0. We have estimated the parameters related to the financial market and interest rate over three different time horizons: (i) January 2nd, 1997 and December 29th, 2006 (1997-2006), (ii) January 3rd, 2007 and December 30th, 2011 (2007-2011), and (iii) January 3rd, 2012 and December 30th, 2016 (2012-2016). The parameters of the risky asset S(t) are obtained from the S&P 500 and the value of constant interest rate r is estimated as the average return of US 3-Month Treasury Bill (on secondary market – daily data). We assume the risk aversion parameter δ is equal to 2 similar to the most common choices of risk aversion parameter in the habit formation and life cycle literature (Munk [2008]; Gong and Li [2006] and Horneff et al. [2015]; Gourinchas and Parker [2002]). We set the subjective discount factor ρ equal to the riskless interest rate r. The estimated parameters along with some assumptions about wealth and preferences are gathered in Tab. II.

We recall the general objective function under the assumption h(t) = 0:



Figure 1: Wealth and consumption under Merton's strategy with different values of the weights (ϕ_c and ϕ_R) in the objective function (31). Dashed lines show the confidence interval (i.e. mean plus and minus two standard deviations). The values of all parameters are estimated for the period 1997-2006 as stated in Tab. II.

$$J(t,R(t)) \equiv \max_{\pi(t)} \mathbb{E}_t \left[\int_t^T \phi_c \frac{(c(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_R \frac{(R(T)-R_m)^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right],$$
(31)

In Fig. 1 to Fig. 5, the solid lines show the mean of 1000 simulations while dashed lines with shaded areas show the confidence interval i.e. mean plus and minus two standard deviations using the parameters estimated for the period 1997-2006. Fig. 6 shows only the mean of 1000 simulations using the parameters estimated for the periods 1997-2006, 2007-2011 and 2012-2016.

Fig. 1 illustrates the sensitivity of the wealth and consumption to the variation in weights assigned to the utility of the terminal wealth and intertemporal consumption in the objective function (31) under Merton's strategy. The graph indicates that weights do affect the optimal portfolio and consumption and a comparatively higher weight must be given to ϕ_R as compared with ϕ_c to maintain wealth above zero in the long run.

Fig. 2 presents the impact of the variation of consumption-wealth ratio on the paths of wealth and consumption for CW strategy, in the absence of subsistence wealth $R_m = 0$. As we can see when y increases, the level of consumption rises but the terminal wealth declines. For further analysis, we consider the case when the subsistence wealth R_m approaches the initial wealth R_0 , the wealth and consumption become less volatile for all values of y, as shown in the Fig. 3. As expected, we can see that a rise or decline in the wealth and consumption in the long run depends on the ratio y.



Figure 2: Wealth and consumption under CW strategy with different values of the consumption-wealth ratio *y*. Dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. II.



Figure 3: Wealth and consumption under CW strategy with different values of the consumption-wealth ratio *y* when $R_m \rightarrow R_0$. Dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. II.



Figure 4: Wealth and consumption under hybrid strategy with different values of the weight ω . Dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. II.

Fig. 4 shows the wealth and consumption with the variation of weights ω for hybrid strategy under the assumption that $R_m = 0$. Interestingly, we can see from the graph that the weight ω for the inflation method must be chosen prudently, the higher the value of ω , the more slowly c(t) converges towards its long term mean and vice versa. Similar to CW strategy, if R_m approaches the initial wealth R_0 , the paths of wealth and consumption becomes less volatile. For this strategy, the initial consumption must not be higher than a certain threshold, otherwise the wealth will decline to zero. In fact, endowment managers face a dilemma in deciding whether to have a spending above a certain level to fund the current activities or to have spending lower than a threshold to achieve the long term objectives. Thus, in choosing the initial level of spending, the management must weigh the potential of greater spending levels in the long run against the need for current spending. The behavior of consumption under hybrid strategy appears to be more acceptable to endowment managers as: (i) it is less volatile in the long run compared to Merton's and CW strategies and (ii) it converges towards its long term mean. In fact, an excessive volatility in spending is undesirable.

Fig. 5 examines the sensitivity of hybrid strategy to the variations in the constant *y*. We see that while *y* increases, so does the volatility in wealth and consumption which implies that *y* must be chosen cautiously for the long term growth of wealth. The path of consumption also depends on the value of *y*. If *y* is higher than the optimal value, then the wealth will not grow in the long run and eventually, the consumption will also decline to a much lower level. If *y* is lower than the optimal level, then the wealth will grow in the long run but the consumption will remain lower even in the long run, provided that the values of other parameters



Figure 5: Wealth and consumption under hybrid strategy with different values of *y*. Dashed lines show the confidence interval. The values of all parameters are estimated for the period 1997-2006 as stated in Tab. II.

remain unchanged.

Fig. 6 shows the comparison of wealth, consumption and risky portfolio (all in monetary units) for Merton's, CW and hybrid strategies using the parameters estimated for three different periods: (i) 1997-2006, (ii) 2007-2011, and (iii) 2012-2016. The graphs of the first column show the dynamic behavior of wealth. We can see that:

- during the first period (1997-2006) under hybrid strategy, the wealth initially declines and then recovers while under Merton's strategy, it declines sharply.
- during the second period (2007-2011), wealth declines for all strategies but the magnitude of decline is far less for hybrid strategy compared to the other two strategies.
- during the third period (2012-2016), wealth rises for all three strategies but it rises less strongly for hybrid strategy.

The graphs of the second column show the dynamic behavior of consumption. For the comparison of consumption we chose, the consumption-wealth ratio for CW strategy and initial consumption for hybrid strategy to match the initial optimal consumption under Merton's strategy. We can see that:

• during the first period, consumption rises under Merton's strategy while it declines under hybrid strategy in a short run and converges towards its long term mean.



Figure 6: Comparison of Merton's strategy with $\phi_R = 0.98$ and $\phi_c = 0.02$ (continuous line), CW strategy with y = 0.04 (dashed line) and hybrid strategy with $\omega = 0.8$ and y = 0.04 (dotted line). The values of all parameters are estimated for the periods 1997-2006, 2007-2011, and 2012-2016 as stated in Tab. II.

- during the second period, consumption remains constant under Merton's strategy, however it declines under the other two strategies.
- during the third period, consumption increases greatly for CW and Merton's strategy while it increases with less intensity for hybrid strategy.

The third column graphs shows the risky portfolio. We can see that during the second period it is optimal to short sell the risky asset whereas in the third period it is optimal to short sell the riskless asset and invest in the risky asset.

6 Conclusion

This paper has provided a brief overview of different spending rules applied by endowment funds. The endowment fund managers adopt these rules to effectively preserve the corpus of the fund and have a stable spending stream. We have obtained the optimal investment strategy under consumption-wealth ratio and hybrid spending rules. Furthermore, we have compared these optimal portfolios and defined spending rules with the classical Merton's optimal portfolio and consumption. We have found that the optimal Merton's portfolio is less riskier than that under consumption-wealth ratio rule, while the Merton's optimal consumption can be replicated using consumption-wealth rule by a suitable selection of the consumption-wealth ratio. The hybrid strategy, for some values of constant parameters, is less risky than both Merton's and consumption-wealth ratio, and consumption under this strategy is less volatile compared to other strategies. The unique characteristic of hybrid rule is that it allow fluctuation in spending during the short run. Also, it converges towards its long term mean regardless of the initial allocation for spending. However, hybrid strategy comparatively outperforms the conventional Merton's strategy and CW strategy when the market is highly volatile but under-performs them when there is a low volatility. Thus, an endowment fund must evaluate, review and modify its spending rule and investment policy periodically, depending on the conditions of the financial market.

Appendix A: Mathematical proofs

For simplification of the notation, we will use the following definitions throughout this appendix:

$$\frac{\partial A(t)}{\partial t} \equiv A_t, \quad \frac{\partial B(t,c(t))}{\partial t} \equiv B_t, \quad \frac{\partial B(t,c(t))}{\partial c} \equiv B_c, \quad \frac{\partial^2 B(t,c(t))}{\partial^2 c} \equiv B_{cc}.$$

A.1 Proof of proposition 1

The value function or indirect utility function is given by

$$J(t,R(t)) \equiv \max_{c(t),\theta_{S}(t)} \mathbb{E}_{t} \left[\int_{t}^{T} \phi_{c} \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} e^{-\rho(s-t)} ds + \phi_{R} \frac{(R(T)-R_{m})^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$

We know that the above objective function must solve the following differential equation (so-called Hamilton-Jacobi-Bellman HJB equation):

$$0 = \max_{c(t),\theta_{S}(t)} \left\{ \begin{array}{c} \phi_{c} \frac{(c(s)-h(s))^{1-\delta}}{1-\delta} - \rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial h}(\alpha(t)c(t) - \beta(t)h(t)) + \frac{\partial J}{\partial R}(R(t)r(t)) \\ + \theta_{S}(t)S(t)(\mu - r(t)) - c(t)) + \frac{1}{2}\frac{\partial^{2}J}{\partial R^{2}}\theta_{S}(t)^{2}S(t)^{2}\sigma^{2} \end{array} \right\}.$$
(32)

The HJB equation (32) in J(t, R(t)), needs a boundary condition so that the value function coincide with the final utility function at the time T: J(T, R(T)) = U(R(T)).

We assume the following guess function

$$J(t, R(t)) = A(t)^{\delta} \frac{(R(t) - h(t)B(t))^{1-\delta}}{1-\delta},$$
(33)

where A(t) and B(t) are the functions that must be determined to solve equation (32). Both functions must satisfy boundary conditions as follows:

$$A(T)^{\delta} = \phi_R \Rightarrow A(T) = \phi_R^{\frac{1}{\delta}}, \quad B(T) = R_m,$$

and the first order conditions (FOCs) of (32) w.r.t. $\theta_S(t)$ and c(t) are:

$$\theta_{S}(t)^{*} = -\frac{\mu - r(t)}{S(t)\sigma^{2}} \frac{\frac{\partial J}{\partial R}}{\frac{\partial^{2} J}{\partial R^{2}}},$$
(34)

$$\phi_c \left(c(s) - h(s) \right)^{-\delta} = \frac{\partial J}{\partial R} - \frac{\partial J}{\partial h} \alpha(t).$$
(35)

By substituting the derivatives of the guess function into both the optimal consumption (35) and the

optimal portfolio (34), we obtain

$$c(t)^{*} = h(t) + \phi_{c}^{\frac{1}{\delta}} \frac{(R(t) - h(t)B(t))(1 + B(t)\alpha(t))^{-\frac{1}{\delta}}}{A(t)},$$
(36)

$$\theta_{S}(t)^{*} = \frac{\mu - r(t)}{S(t)\sigma^{2}} \frac{R(t) - h(t)B(t)}{\delta}.$$
(37)

Inserting the optimal consumption (36), the optimal portfolio (37) and the partial derivatives of the guess function into equation (32), we have

$$\begin{split} 0 &= \frac{\delta}{1-\delta} \phi_c^{\frac{1}{\delta}} \left(1+B(t)\alpha(t)\right)^{1-\frac{1}{\delta}} - \frac{\rho A(t)}{1-\delta} + \frac{\delta}{1-\delta} A_t - \frac{A(t)h(t)B_t}{R(t)-h(t)B(t)} \\ &- \frac{A(t)B(t)\alpha(t)h(t)}{R(t)-h(t)B(t)} + \frac{A(t)B(t)\beta(t)h(t)}{R(t)-h(t)B(t)} + r(t)A(t) \\ &+ \frac{A(t)h(t)B(t)r(t)}{R(t)-h(t)B(t)} + A(t)\frac{(\mu-r(t))^2}{2\sigma^2\delta} - \frac{A(t)h(t)}{R(t)-h(t)B(t)}, \end{split}$$

which can be separated into two differential equations, one that consists of the terms containing $(R(t) - h(t)B(t))^{-1}$ and one without them and after few simplifications we have

$$\begin{cases} 0 = \phi_c^{\frac{1}{\delta}} \left(1 + B(t)\alpha(t) \right)^{\frac{\delta-1}{\delta}} + A_t + A(t) \left(\frac{1-\delta}{\delta} r(t) - \frac{\rho}{\delta} + \frac{(1-\delta)(\mu - r(t))^2}{2\delta^2 \sigma^2} \right) \\ 0 = B_t + B(t) \left(\alpha(t) - \beta(t) - r(t) \right) + 1 \end{cases}$$
(38)

The above ordinary differential equations, together with their corresponding boundary conditions, have the following unique solutions:

$$A(t) = \phi_R^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta - 1}{\delta}r(u) + \frac{\rho}{\delta} + \frac{(\delta - 1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} + \int_t^T \phi_c^{\frac{1}{\delta}} \left(1 + B(s)\alpha(s)\right)^{\frac{\delta - 1}{\delta}} e^{-\int_t^s \left(\frac{\delta - 1}{\delta}r(u) + \frac{\rho}{\delta} + \frac{(\delta - 1)(\mu - r(u))^2}{2\delta^2 \sigma^2}\right) du} ds, B(t) = R_m e^{-\int_t^T (-\alpha(u) + \beta(u) + r(u)) du} + \int_t^T e^{-\int_t^s (-\alpha(u) + \beta(u) + r(u)) du} ds.$$

The subsistence level case, i.e. h(t) = h and the classical case, i.e. h(t) = 0 can be easily followed from the above results.

A.2 Proof of proposition 2

The value function is given by

$$J(t,R(t)) \equiv \max_{\theta_{S}(t)} \mathbb{E}_{t} \left[\frac{(R(T)-R_{m})^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$

For this objective function, we can write the following HJB equation

$$0 = \max_{\theta_{S}(t)} \left\{ \begin{array}{c} -\rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R} \left(R(t)r(t) + \theta_{S}(t)S(t)\left(\mu - r(t)\right) - c(t) \right) \\ + \frac{1}{2} \frac{\partial^{2} J}{\partial R^{2}} \theta_{S}(t)^{2}S(t)^{2}\sigma^{2} \end{array} \right\},$$
(39)

and the first order condition (FOC) of (39) w.r.t. $\theta_S(t)$ is

$$\theta_{S}(t)^{*} = -\frac{\mu - r(t)}{S(t)\sigma^{2}} \frac{\frac{\partial J}{\partial R}}{\frac{\partial^{2} J}{\partial R^{2}}}.$$
(40)

We assume the following guess function

$$J(t, R(t)) = A(t)^{\delta} \frac{(R(t) - B(t))^{1 - \delta}}{1 - \delta},$$
(41)

where A(t) is the function that must solve equation (39), with the boundary condition A(T) = 1, while B(t) must satisfy the boundary condition $B(T) = R_m$, and the optimal portfolio process $\theta_S(t)^*$ in (40), for our guess function can be written as

$$\theta_{S}(t)^{*} = \frac{\mu - r(t)}{S(t)\sigma^{2}} \frac{R(t) - B(t)}{\delta}.$$
(42)

Substituting the optimal portfolio (42) and partial derivatives of the guess function into (39), we get

$$0 = \frac{\delta}{1-\delta}A_t + A(t)\left(r(t) - y - \frac{\rho}{1-\delta} + \frac{1}{2}\frac{(\mu - r(t))^2}{\sigma^2\delta}\right) + \frac{A(t)}{R(t) - B(t)}\left(B(t)(r(t) - y) - B_t\right),$$

which can be separated into two differential equations, one that consists of the terms containing $(R(t) - B(t))^{-1}$ and one without them and after few simplifications, we have

$$\begin{cases} 0 = A_t + A(t) \left(\frac{1-\delta}{\delta} \left(r(t) - y \right) + \frac{1-\delta}{2\delta^2} \frac{(\mu - r(t))^2}{\sigma^2} - \frac{\rho}{\delta} \right), \\ 0 = B(t) \left(r(t) - y \right) - B_t. \end{cases}$$

The above ordinary differential equations with their corresponding boundary conditions have the following solutions:

$$A(t) = e^{-\int_t^T \left(\frac{\delta-1}{\delta}r(s) - \frac{\delta-1}{\delta}y + \frac{\delta-1}{2\delta^2}\frac{(\mu-r(s))^2}{\sigma^2} - \frac{\rho}{\delta}\right)ds}, \quad B(t) = R_m e^{-\int_t^T (r(s)-y)ds}.$$

A.3 Proof of proposition 3

The value function is given by

$$J(t,R(t),c(t)) \equiv \max_{\theta_{S}(t)} \mathbb{E}_{t} \left[\frac{(R(T)-R_{m})^{1-\delta}}{1-\delta} e^{-\rho(T-t)} \right].$$

For this objective function, we can write the following HJB equation

$$0 = \max_{\theta_{S}(t)} \left\{ \begin{array}{c} -\rho J + \frac{\partial J}{\partial t} + \frac{\partial J}{\partial R} \left(R(t)r(t) + \theta_{S}(t)S(t)(\mu - r(t)) - c(t) \right) \\ + \frac{\partial J}{\partial c} \left(1 - \omega \right) \left(1 + y \right) \left(\frac{yR(t)(1 + r(t)) + y\theta_{S}(t)S(t)(\mu - r(t))}{1 + y} - c(t) \right) \\ + \frac{1}{2} \frac{\partial^{2} J}{\partial R^{2}} \theta_{S}(t)^{2} S(t)^{2} \sigma^{2} + \frac{1}{2} \frac{\partial^{2} J}{\partial c^{2}} a^{2} \theta_{S}(t)^{2} S(t)^{2} \sigma^{2} + \frac{\partial^{2} J}{\partial c \partial R} a \theta_{S}(t)^{2} S(t)^{2} \sigma^{2} \right\},$$
(43)

and the first order condition (FOC) of (43) w.r.t. $\theta_S(t)$ is

$$\boldsymbol{\theta}_{S}(t)^{*} = -\frac{\boldsymbol{\mu} - \boldsymbol{r}(t)}{S(t)\boldsymbol{\sigma}^{2}} \frac{\frac{\partial J}{\partial R} + \frac{\partial J}{\partial c}a}{\frac{\partial^{2}J}{\partial R^{2}} + \frac{\partial^{2}J}{\partial c^{2}}a^{2} + 2\frac{\partial^{2}J}{\partial c\partial R}a}.$$
(44)

We assume the following guess function

$$J(t, R(t), c(t)) = A(t)^{\delta} \frac{(R(t) - B(t, c(t))^{1 - \delta}}{1 - \delta}.$$
(45)

Thus, the optimal portfolio process $\theta_S(t)^*$ in (44), can be written as

$$\theta_S(t)^* = -\frac{\mu - r(t)}{S(t)\sigma^2} A(t)^{\delta} \left(R(t) - B(t, c(t)) \right)^{-\delta} \frac{1 - B_c a}{F(t, c(t))},$$

where we define

$$F(t,c(t)) = -\delta A(t)^{\delta} (R(t) - B(t,c(t)))^{-\delta-1} - \delta A(t)^{\delta} (R(t) - B(t,c(t)))^{-\delta-1} B_c^2 a^2 - A(t)^{\delta} (R(t) - B(t,c(t)))^{-\delta} B_{cc} a^2 + 2\delta A(t)^{\delta} (R(t) - B(t,c(t)))^{-\delta-1} B_c a^2 - A(t)^{\delta} (R(t) - B(t,c(t)))^{\delta-1} B_c a^2 - A(t)^{\delta} (R(t) - A(t)^{\delta} (R(t) - B(t,c(t)))^{-\delta-1} B_c a^2 - A(t)^{\delta} (R(t) - B(t,c(t)))^{\delta-1} B_c a^2 - A(t)^{\delta} (R(t) - B(t,c(t)))^{\delta-1} B_c a^2 - A(t)^{\delta} (R(t) - A(t)^{\delta} (R(t) - A(t)^{\delta} (R(t) - A(t)^{\delta} (R(t) - A(t)^{\delta} (R(t)))^{\delta-1} B_c a^2 - A(t)^{\delta} (R(t) - A(t)^{\delta} (R($$

The HJB equation (43), under the hypotheses $B_c = \eta$ and $B_{cc} = 0$, becomes

$$\begin{split} 0 &= -A(t)\frac{\rho}{1-\delta} + \frac{\delta}{1-\delta}A_t - A(t)\frac{B_t}{R(t) - B(t,c(t))} + A(t)r + A(t)\frac{rB(t,c(t))}{R(t) - B(t,c(t))} \\ &- A(t)\frac{c(t)}{R(t) - B(t,c(t))} - A(t)\eta a(1+r) - A(t)\frac{\eta a(1+r)B(t,c(t))}{R(t) - B(t,c(t))} \\ &+ A(t)\frac{\eta (1-\omega)c(t)}{R(t) - B(t,c(t))} + A(t)\frac{\eta ac(t)}{R(t) - B(t,c(t))} + \frac{A(t)(\mu - r)^2}{2\sigma^2\delta}, \end{split}$$

which can be separated into two differential equations, one that consists of the terms containing $(R(t) - B(t, c(t)))^{-1}$ and one without them and after few simplifications, we have

$$\begin{cases} 0 = A_t + A(t) \left(\frac{1-\delta}{\delta} r - \frac{\rho}{\delta} - \frac{1-\delta}{\delta} \eta a(1+r) + \frac{1-\delta}{\delta} \frac{(\mu-r)^2}{2\sigma^2 \delta} \right), \\ 0 = B_t - B(t, c(t)) \left(r - \eta a(1+r) \right) + c(t) \left(1 - \eta \left(1 - \omega \right) - \eta a \right), \end{cases}$$
(46)

and the optimal portfolio in this case can be written as using (44)

$$\theta_{S}(t)^{*} = \frac{\mu - r}{\delta S(t)\sigma^{2}} \frac{R(t) - B(t, c(t))}{1 - \eta a}.$$

These equations are both ordinary linear differential equations and their boundary conditions can be obtained from the boundary condition of the HJB equation: A(T,c(T)) = 1 and $B(T,c(T)) = R_m$.

The solution of the ODE (46), together with their boundary conditions is given by

$$A(t) = e^{-\left(\frac{\delta-1}{\delta}r + \frac{\rho}{\delta} - \frac{\delta-1}{\delta}\eta a(1+r) + \frac{\delta-1}{\delta}\frac{(\mu-r)^2}{2\sigma^2\delta}\right)(T-t)}.$$

Since the second differential equation has been obtained under the hypothesis that $B_c = \eta$ and $B_{cc} = 0$,

then the only consistent functional form for B(t, c(t)) is

$$B(t,c(t)) = \eta(t)c(t) + h(t),$$

where $\eta(t)$ and h(t) may be functions of time. Thus the second ODE can be rewritten as follows

$$\left(\frac{\partial \eta(t)}{\partial t}c(t) + \frac{\partial h(t)}{\partial t}\right) - \left(r - \eta(t)a(1+r)\right)\left(\eta(t)c(t) + h(t)\right) - \left(\eta(t)(1 - \omega + a) - 1\right)c(t) = 0,$$

which can be separated into two ODE's, one which contains c(t) and one which contains all the terms without c(t)

$$0 = \frac{\partial \eta(t)}{\partial t} - (r - \eta(t) a(1 + r)) \eta(t) - (\eta(t)(1 - \omega + a) - 1), \qquad (47)$$

$$0 = \frac{\partial h(t)}{\partial t} - (r - \eta(t) a(1+r)) h(t), \qquad (48)$$

with the boundary condition

$$\eta(T)c(T) + h(T) = R_m,$$

as $B(T, c(T)) = R_m$ and consequently $\eta(T) = 0$ and $h(T) = R_m$, which means that η must be a constant and the ODE (47) (with $\frac{\partial \eta(t)}{\partial t} = 0$) gives

$$\eta^{2}a(1+r) - \eta(1+r+a-\omega) + 1 = 0,$$

which has two solutions:

$$\eta = \frac{(1 + r + a - \omega) \pm \sqrt{(1 + r + a - \omega)^2 - 4a(1 + r)}}{2a(1 + r)},$$

where one of the two solutions (η^*) must be chosen suitably.

The ODE (48), together with the boundary condition, has a unique solution

$$h(t) = R_m e^{-(r-\eta^* a(1+r))(T-t)}.$$

Therefore,

$$B(t,c(t)) = \eta^* c(t) + R_m e^{-(r-\eta^* a(1+r))(T-t)},$$
(49)

such that $-\infty < \eta^* < \frac{1}{a}$.

References

- Isabelle Bajeux-Besnainou and Kurtay Ogunc. Spending rules for endowment funds: A dynamic model with subsistence levels. *Rev Quant Finan Acc*, 27:93–107, 2006.
- Georg Cejnek, Richard Franz, Otto Randl, and Neal Stoughton. A survey of university endowment management research. *Journal of Investment Management*, Third Quarter:40, 2014.
- George M. Constantinides. Habit formation: A resolution of the equity premium puzzle. *The Journal of Political Economy*, 98(3):519–543, 1990.
- Stephen G. Dimmock. Background risk and university endowment funds. *The Review of Economics and Statistics*, 94(3):789–799, 2012.
- Philip H. Dybvig. Using asset allocation to protect spending. *Financial Analysts Journal*, 55(1):49–62, 1999.
- Richard M. Ennis and J. Peter Williamson. *Spending policy for educational endowments*. The Common Fund, 1976.
- Steve P. Fraser and William W. Jennings. Examining the use of investment policy statements. *The Journal of Wealth Management*, 13(2):10–22, 2010.
- Ning Gong and Tao Li. Role of index bonds in an optimal dynamic asset allocation model with real subsistence consumption. *Applied Mathematics and Computation*, 174:710–731, 2006.
- Pierre-Olivier Gourinchas and Jonathan A. Parker. Consumption over the life cycle. *Econometrica*, 70(1): 47–89, 2002.
- Henry Hansmann. Why do universities have endowments? The Journal of Legal Studies, 19(1):3-42, 1990.

- Vanya Horneff, Raimond Maurer, Olivia S. Mitchell, and Ralph Rogalla. Optimal life cycle portfolio choice with variable annuities offering liquidity and investment downside protection. *Insurance: Mathematics* and Economics, 63(Supplement C):91–107, 2015.
- Roger T. Kaufman and Geoffrey Woglom. Modifying endowment spending rules: Is it the cure for overspending? *Journal of Education Finance*, 31(2):146–171, 2005.
- Robert C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.
- Robert C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic theory*, 3(4):373–413, 1971.
- Robert C. Merton. *Optimal Investment Strategies for University Endowment Funds*. University of Chicago Press, continuous time finance edition, 1993.
- Claus Munk. Portfolio and consumption choice with stochastic investment opportunities and habit formation in preferences. *Journal of Economic Dynamics & Control*, 32:3560–3589, 2008.
- NACUBO. Nacubo-commonfound study of endowments (ncse). Technical report, National Association of College and University Business Officers, 2016.
- Paul A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51(3):239–246, 1969.
- Verne O. Sedlacek and William F. Jarvis. Endowment spending: Building a stronger policy framework. *Commonfund Institute*, page 24, 2010.
- James Tobin. What is permanent endowment income? *The American Economic Review*, 64(2):427–432, 1974.