Stochastic Processes in Credit Risk Modelling

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Abstract
In credit risk modelling, jump processes are widely used to describe both default and rating migration events. This work is mainly a review of some basic definitions and properties of the jump processes intended for a preliminary step before more advanced lectures on credit risk modelling. We focus on the Poisson process and some generalisations, like the compounded and the double stochastic Poisson processes, which are widely used for describing the time-inhomogeneous dynamic either of the default process or of the credit rating transition. As such, much of the material is not new, but focused and organized from a credit risk perspective. Moreover it contains detailed proofs of some fundamental results. Other original contributions come from examples and simulated studies, which help the reader to better understand the features of the described processes.

Keywords: Markov process, Jump process, Poisson process, Cox process, Doubly stochastic Poisson process.

MSC: 60G55, 60J75, 91B28.

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1 Introduction

In this work we survey definitions and basic properties of some stochastic processes involved in credit risk modelling. We mainly focus on point processes which are widely used in risk theory and which play now an important role also in the actuarial and reduced form framework to credit risk modelling. These processes are used in many disciplines, like physics, biology, image analysis, operations research (queueing theory), hydrology (see Calenda, Napolitano (1999)), volcanology (see Jaquet, Carniel (2001), Jaquet et al. (2000)), etc., in order to describe the random occurrence of discrete value events over both time and space. In credit risk models point processes are suitable to capture the discontinuity induced by a default or credit migration event and play an important role in both defaultable claim pricing and credit portfolio loss evaluation.

Point stochastic processes have already been applied to many other financial problems like: actuarial or insurance mathematics (see Grandell (1991)), Extreme Value Theory (see Embrechts, Kluepperlberg and Mikosch (1997)), stochastic volatility and term structure models. Finally note that another fundamental process involved in credit risk models is the traditional division process which is not treated in detail in this work. We refer the interested reader to the Karatzas and Shreve (1988), Revuz and Yor (1999).

In more detail point processes have been widely used in the recent credit risk literature on both structural and reduced form approach, but in the latter they play a crucial role. For an application within the structural approach, see Zhou (1997). For their use within the reduced form approach see Lando (1998) and Jarrow, Lando and Turnbull (1997). In the credit risk theory the occurrence of the default event is of interest because it is a source of risk that affects the current value of the defaultable securities. In the reduced form model the default of a firm is described through a point process. Furthermore the intensity of the process can be made dependent on a set of exogenous state variables. This particular reduced form approach to credit risk evaluation, is called intensity based.

In early papers on the intensity based approach (see Jarrow, Turnbull (1995)) the default event is modelled as the first jump of a Poisson process, which is assumed to be independent of the other financial variables in the model. In more recent contributions, some generalisations of the Poisson process have been introduced. In particular Lando (1998) proposes to use a Cox process (or doubly stochastic Poisson process, see Grandell (1976), (1991)), which allows for a time-inhomogeneous dynamic of the stochastic intensity driving the default. Moreover the intensity is made dependent on a set of variables, like macroeconomic factors or financial market variables.
A possible criticism to this framework, when we consider a single credit position, is that in the underlying probability model more than one default may occur. In practice, within the reduced form models, only the first jump of the default process is of interest. Another way to overcome this limit, in discrete time, is to apply duration models (see Gouriéroux and Monfort (2002)). When we consider a portfolio of homogeneous credit positions the jump process represents the multiple defaults event occurring in the portfolio and represent a powerful modelling tool.

The work is structured as follows. In Section 2, following Bouleau (2000), we introduce some basic definitions and properties of a Markov process. In Section 3 some Markov point processes, like homogeneous and compounded Poisson processes and their associated martingales, are defined. In particular the Poisson counting model is defined also through a differential argument, which is particularly useful in order to understand the relation between the counting process, the inter-arrival distribution and the waiting time distribution. In Section 4 we treat in more detail the Poisson stochastic process. It represents a simple, but meaningful probability model, that can also be easily extended towards more "realistic" stochastic models. Section 5 gives a more general definition of point process and Section 6 concludes.

2 Markov jump process

In this section we show that Markov jump processes are Markov processes where the time length of the permanence in each state is described by a random variable. In credit risk modelling, the p.d.f. of this r.v. is strictly related to the probability of default. Let us introduce a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a state space \((E, \mathcal{E})\). If \(E\) is assumed to be countable then a continuous time Markov process defines as follows.

**Definition 2.1.** (Markov Process) - A stochastic process \((X_t)_{t\geq 0}\) with values in \((E, \mathcal{E})\), is a Markov process if \(\forall t, u \geq 0\) and \(\forall A \in \mathcal{E}\), \(\mathbb{P}\{X_{t+u} \in A \mid X_s, s \leq t\} = \mathbb{P}\{X_{t+u} \in A \mid X_t\}, \mathbb{P}\text{-a.s.}\).

If \(E\) is a finite countable set then \(P_{s,t}\) is a stochastic matrix defined as follows\(^1\):

\[
P_{s,t} = \begin{cases} 
\mathbb{P}\{X_t = y \mid X_s = x\} & \text{if } \mathbb{P}\{X_s = x\} \neq 0 \\
\mathbb{I}\{x=y\} & \text{if } \mathbb{P}\{X_s = x\} = 0
\end{cases}
\]

\(^1\)We denote with \(\mathbb{I}\{x=y\}\) the indicator function, which is defined as: \(\mathbb{I}\{x=y\} = \{1 \text{ if } x = y, 0 \text{ otherwise}\}\)
which satisfies the Chapman-Kolmogorov equation:

\[
\mathbb{P} \{ X_u = z \mid X_s = x \} = \\
\sum_{y \in \mathcal{E}} \mathbb{P} \{ X_t = y \mid X_s = x \} \mathbb{P} \{ X_u = z \mid X_t = y \} \tag{2}
\]

\[
= \sum_{y \in \mathcal{E}} P_{s,t}(x,y) P_{t,u}(y,z) \quad \Leftrightarrow \quad P_{s,u}(x,z) = P_{s,t} P_{t,u} \tag{3}
\]

Now, let us assume the state space \( \mathcal{E} \) is not countable and let \( \mathcal{E}_b \) be the set of all bounded \( \mathcal{E} \)-measurable functions with values in \( \mathbb{R} \), then the definition of Markov process modifies as follow.

**Definition 2.2. (Markov Process)** - Given a state space \((\mathcal{E}, \mathcal{E})\) with its Borel \( \sigma \)-algebra, and a family of transition probabilities \((P_{s,t})_{0 < s < t}\) on \((\mathcal{E}, \mathcal{E})\) such that \( \forall s < t < u \), \( P_{s,u} = P_{st} P_{t,u} \), and a stochastic process \((X_t)_{t \geq 0}\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\), then \((\Omega, \mathcal{F}, \mathbb{P}, (X_t)_{t \geq 0})\) is a Markov process, with transition probability \( P_{s,t} \) if \( \forall f \in \mathcal{E}_b, \forall s < t \):

\[
\mathbb{E} (f(X_t) \mid X_u, u < s) = P_{s,t} f(X_s) \\
= \int_{x \in \mathcal{E}} f(x) P_{s,t}(X_s, dx) = \mathbb{E} (f(X_t) \mid X_s) \quad \mathbb{P} - a.s.. \tag{5}
\]

If the process has an initial distribution \( \pi \), then \( P_\pi \) and \( \mathbb{E}_\pi(\cdot) \) denote the transition probability and the expected value, respectively. If the initial distribution is a Dirac mass \( \delta_x \), the previous notation becomes \( P_x \) and \( \mathbb{E}_x(\cdot) \). The Markov process is said to be *homogeneous* if:

\[
P_{s,t} = P_{s+h,t+h}, \quad \forall s < t, \forall h > 0 \tag{6}
\]

and we will write: \( P_{0,t} = P_t \) and \( P_{s,t} = P_{t-s} \). Note that the transition probability defines a semi-group operator:

\[
P_{t} P_{s} = P_{s+t}, \quad \forall s, t > 0 \tag{7}
\]

**Definition 2.3. (Jump process)** - Given a transition semi-group operator \( P_t \) defined on the space \((\mathcal{E}, \mathcal{E})\), a Markov process with transition probability \( P_t \) is a Jump process if \( \forall \pi \), initial probability, almost every trajectory is right continuous and constant, except in a countable set of jumps.

Let \( \mathcal{F}_t = \sigma(X_s, s \leq t) \) a sub \( \sigma \)-algebra of \( \mathcal{F} \). Then from previous definitions, the following properties hold:

1. \( \mathbb{E}_x(f(X_t)) = P_t f(x), \forall f \in \mathcal{E}_b \) and \( P_x(X_t \in A) = P_t(x, A), \forall A \in \mathcal{E} \)
2. \( \mathbb{E}_\pi (f(X_t) \mid F_s) = \mathbb{E}_\pi (f(X_t) \mid X_s) = P_{t-s} f(X_s) = \mathbb{E}_{X_s} (f(X_{t-s})) \mathbb{P}_\pi \)-a.s.

3. Let \( f_0, \ldots, f_n \) be a sequence of functions in \( \mathcal{E}_b \) then \( \forall t_0 < t_1 < \ldots < t_n \):
\[
\mathbb{E}_\pi (f_0(X_{t_0}) \ldots f_n(X_{t_n})) = \int_{x_0 \in E} \pi(dx_0) f(x_0) \int_{x_1 \in E} P_{t_1} (x_0, dx_1) f_1(x_1) \int_{x_2 \in E} P_{t_2-t_1} (x_1, dx_2) f_2(x_2) \ldots \int_{x_n \in E} P_{t_n-t_{n-1}} (x_{n-1}, dx_n) f_n(x_n) \tag{8}\]

**Theorem 2.1. (Simple Markov property)** - Let \( (X_t)_{t \geq 0} \) be a Markov jump process and \( A \) a measurable set of trajectories, that is \( A \subset \Omega = \mathbb{E}^{\mathbb{R}_+} \) and \( A \in \mathcal{F} = \mathcal{E}^{\otimes \mathbb{R}_+} \). The trajectories \( X_{s+t} \) (which stays for \( s \rightarrow X_{s+t} \)) satisfy the following property: \( P_\pi (X_{s+t} \in A \mid \mathcal{F}_t) = P_{X_t} (X \in A), \mathbb{P}_\pi \text{-a.s.} \).

Before introducing the strong Markov property we need to define a stopping time and some of its properties. Stopping times widely apply to credit risk modelling. An example is the definition of the default or rating transition events as the first time an underlying stochastic process (the firm’s value) crosses an exogenously fixed barrier.

**Definition 2.4. (Stopping time)** - A random variable \( \tau : \Omega \rightarrow \mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\} \) is called stopping time if \( \tau \leq t \) \( \in \mathcal{F}_t \), \( \forall t \).

The first entrance time, defined as \( D = \inf \{t \geq 0 : X_t \in F \} \), where \( F \subset E \), is an example of stopping time. Let \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) be a filtration, then a stopping time \( \tau \) is \( \mathcal{F} \)-predictable if there exists an increasing sequence of stopping time \( \{\tau_n\}_{n \geq 0} \) such that \( \tau_n < \tau \) and \( \lim \tau_n = \tau \). A stopping time \( \tau \) is called \( \mathcal{F} \)-inaccessible if for any \( \mathcal{F} \)-predictable stopping time \( S \), \( \mathbb{P} \{\omega \in \Omega \mid \tau (\omega) = S (\omega)\} = 0 \).

**Definition 2.5. (Stopping time \( \sigma \)-algebra)** - Given a stopping time \( \tau \) the associated \( \sigma \)-algebra is defined as follows: \( \mathcal{F}_\tau = \{ A \in \mathcal{E}^{\otimes \mathbb{R}_+} \mid A \cap \{ \tau \leq t \} \in \mathcal{F}_t \}, \forall t \} \).

**Theorem 2.2.** Let \( \tau \) be a stopping time, then \( \mathcal{F}_\tau \) is a \( \sigma \)-algebra.

**Theorem 2.3.** Let \( \tau \) be a stopping time and \( (X_t)_t \) a Markov process, then \( X_\tau \) is \( \mathcal{F}_\tau \)-measurable.

From previous definitions and theorems, the following property holds for a Markov jump process.
Theorem 2.4. (Strong Markov property) - A jump process $(X_t)_{t \geq 0}$ is strongly Markovian. This means that $\forall \tau, \mathcal{F}_\tau$-stopping time and $\forall A \in \mathcal{E}_{[0,\infty]}$:

$$P \{ \{ X_{+\tau} \in A \} \cap \{ \tau < \infty \} \mid \mathcal{F}_\tau \} = \mathbb{I}_{\{\tau < \infty\}} P_{X_\tau} (\{X \in A\}). \tag{9}$$

The following result shows that in a jump process with initial value $x$, the waiting time for the next jump is an exponentially distributed random time. Furthermore the new value of the process at the instant of the jump $(X_{\tau})$ is independent to the random time of the jump.

Theorem 2.5. Let $\tau = \tau(x)^c$ be the first enter time in the complementary set of $\{x\}$, with $x \in E$, then: (a) $P_x (\{\tau > s\}) = e^{-\lambda(x)s}$, where $\lambda(x)$ is a positive or null function defined on $E$; (b) under the probability $P_x$, if $\lambda(x) \neq 0$ then $X_\tau$ and $\tau$ are independent.

The function $\lambda(x)$ is called intensity of the process. In the following we assume: $\| \lambda(x) \|_{\infty} = \sup_{x \in E} |\lambda(x)| < \infty$. Under the previous assumption and from Theorem 2.5 the expected value of the stopping time is: $E_x (T_{\{x\}^c}) = (\lambda(x))^{-1}$.

Theorem 2.6. - Let $(X_t)$ be a jump process with bounded intensity $\lambda(x)$. Then it exists a stochastic matrix $P$ such that the transition semi-group of the process can be written as: $P_t = \exp (tI_{\lambda}(P-I))$, where $I_{\lambda}$ is the by intensity function product operator. Furthermore the matrix $P$ is defined as follows:

$$P(x,y) \begin{cases} P_x (X_{T_x}^c = y) & \text{if } \lambda(x) \neq 0 \\ \mathbb{I}_{\{x=y\}} & \text{if } \lambda(x) = 0 \end{cases} \tag{10}$$

and in a neighbor of $t = 0$: $P_x (X_t = x) = 1 - \lambda(x)t + \epsilon_1 (x,y)t$ and $P_x (X_t \in A) = \lambda(x)t + P (x,A) + te_2 (x,t,A)$, $\forall A \subseteq \{x\}^c$, where $\epsilon_1 (x,t)$ and $e_2 (x,t,A)$ converge towards zero, uniformly in $x$ and in $A$, when $t \to 0$.

In the following of the section we show a general method for building a Markov jump process. This method allows us to completely specify the jump process by means of a transition probability $P$ and of an intensity function $\lambda(x)$. We require this function to be bounded, but it is possible to define the jump process also with an unbounded intensity (see Bouleau (2000) for more details). Theorem 2.7 will be useful also in the definition of Poisson process (see next section).

Theorem 2.7. (Markov jump process) - Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, assume $(Z_n)_{n \geq 0}$ and $(\tau_n (x))_{n \geq 1}$, with $x \in E$, are two sequences of random variables such that:
1. $Z_0, Z_1, \ldots, Z_n, \ldots$ is a Markov chain with transition matrix $P$ and initial probability $\pi$;

2. $\forall x \in E$, $\tau_n(x)$ is a variable such that $P(\{\tau_n(x) > t\}) = \lambda(x) e^{-\lambda(x)t}$, $\forall t \geq 0$;

3. $\tau_n(x)$ is a measurable function of $(\omega, x)$;

4. the processes $(Z_n)_{n \geq 0}$, $(\tau_1(x))_{x \in E}$, $(\tau_2(x))_{x \in E}$, $\ldots$ are independent;

then the following series:

$$X_t = \begin{cases} Z_0 \mathbb{1}_{\{t < \tau_1(Z_0)\}} + Z_1 \mathbb{1}_{\{\tau_1(Z_0) \leq t < \tau_1(Z_0) + \tau_2(Z_1)\}} + \cdots + \\ + Z_n \mathbb{1}_{\{\tau_1(Z_0) + \cdots + \tau_n(Z_{n-1}) \leq t < \tau_1(Z_0) + \cdots + \tau_n + 1(Z_n)\}} + \cdots 
\end{cases} \quad (11)$$

defines a jump process with initial probability measure $\pi$ and transition semi-group $P_t = e^{tI_\lambda(P-I)}$.

By removing the hypothesis of exponentially distributed random time, it is possible to extend the definition of Markov jump process toward the definition of a semi-Markov jump process (see Dacunha-Castelle, Duflo (1993) for further details).

### 3 Poisson Process

It is possible to define a Poisson counting process (or simply Poisson process) as a particular case of Markov counting process (or Markov jump process). Thus all the properties defined in previous paragraphs apply.

**Definition 3.1.** (Poisson counting process) - A Poisson process is a Markov jump process with values in $\mathbb{N}$, constant intensity $\lambda$ and transition matrix $P(n, m) = \mathbb{1}_{(m-n) = 1}$.

This specification of the transition probability makes the process to be "non-decreasing" with transition semi-group: $P_t = e^{t\lambda(P-I)}$. In order to propose some generalizations, it is useful to give another definition of Poisson process, which is equivalent to that one in Definition 3.1.

**Definition 3.2.** (Homogeneous Poisson process) - Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(N_t)_{t \geq 0}$ a stochastic process defined on it. Let $\mathcal{F}_t = \sigma\{N_s, s \leq t\}$ the $\sigma$-field generated by the process, then $(N_t)_{t \geq 0}$ is called homogeneous Poisson process with intensity $\lambda$, if the following two conditions are satisfied:

1. $\forall 0 \leq s < t$ the increment $(N_t - N_s)$ is independent of the $\sigma$-field $\mathcal{F}_s$;
2. \( \forall 0 \leq s < t \) the increment \((N_t - N_s)\) has the following Poisson distribution:

\[
\mathbb{P}\{(N_t - N_s) = k \mid \mathcal{F}_s\} = \mathbb{P}\{(N_t - N_s) = k\} = \frac{\left(\lambda (t - s)\right)^k}{k!} e^{-\lambda (t-s)}
\]

with \( k = 0, 1, 2, \ldots \).

The process is called homogeneous because the increment distribution is invariant with respect to time shifts. In the following we propose some generalisations of the Poisson model and show two alternative derivations of the Poisson stochastic model: through a sum of memoryless process and through a differential argument. These two methods highlight the role of the parameters and random time distributions involved in the probabilistic model, underlying this kind of process.

In the first method the Poisson process is built through the use of a sequence of random times, which describe the arrival phenomena. The key idea follows from a simple observation. Let \( \tau_k = \inf \left\{ t > \tau_{k-1} : (N_t - N_{\tau_{k-1}}) = 1 \right\} \) be a stopping time, then from Definition 3.2, the random variable \( \xi_k = \tau_k - \tau_{k-1} \), which represents the inter-arrival time, is exponentially distributed with parameter \( \lambda \). This suggests that a Poisson process can be constructed also by specifying the inter-arrival time distribution.

**Theorem 3.1.** - Let \( \xi_1, \xi_2, \ldots \) be a sequence of random variables defined on the \((\Omega, \mathcal{F}, \mathbb{P})\) with common exponential distribution with parameter \( \lambda > 0 \). The process \((N_t)_{t \geq 0}\), defined by setting \( N_t = 0 \) if \( t < \xi_1 \) and:

\[
N_t = \sum_{k=1}^{\infty} k \mathbb{I}_{\left\{ \xi_1 + \xi_2 + \ldots + \xi_k \leq t < \xi_2 + \ldots + \xi_{k+1} \right\}}
\]

otherwise, is a Poisson process with intensity \( \lambda \).

**Proof.** - It is sufficient to prove that the Laplace transform of the process \((N_t)_{t \geq 0}\) is equal to \( e^{\lambda (u-1)} \) (see Theorem 3.4), which is the Laplace transform of a Poisson process with constant intensity \( \lambda \). The process counts the number of the events in the interval \([0,t]\). Define \( \tau_i = \xi_1 + \ldots + \xi_i \), then the number of events in the interval is:

\[
N_t = \sum_{k=1}^{\infty} k \mathbb{I}_{\left\{ \xi_1 + \xi_2 + \ldots + \xi_k t < \xi_1 + \xi_2 + \ldots + \xi_{k+1} \right\}}
= \sum_{k=1}^{\infty} k \mathbb{I}_{\left\{ \tau_k \leq t < \tau_{k+1} \right\}} = \sum_{k=1}^{\infty} \mathbb{I}_{[0,t]}(\tau_k)
\]
Therefore the Laplace transform of the process is:

\[
\mathbb{E}(u^N_t) = \mathbb{E} \left( \sum_{k=1}^{\infty} \mathbb{I}_{[0,\xi]}(\tau_k) \right) = \lim_{m \to \infty} \prod_{k=1}^{m} \mathbb{E}(u^\mathbb{I}_{[0,\xi]}(\tau_k)) \\
= \lim_{m \to \infty} \int_0^\infty g(\xi_1)g(\xi_1 + \xi_2) \ldots g(\xi_1 + \ldots + \xi_m) \lambda e^{-\lambda \xi_1} d\xi_1 \ldots \lambda e^{-\lambda \xi_m} d\xi_m
\]

where \( g(\tau_i) = \mathbb{I}_{[0,\xi]}(\tau_i) \). Consider now, the following change of variable: \( v_i = \xi_1 + \ldots + \xi_i \), with \( i = 1, \ldots, m \), which is a bijective transformation on the partition \( v_1 < v_2 < \ldots < v_m \) of \( \mathbb{R}^m_+ \). The integral becomes:

\[
\mathbb{E}(u^N_t) = \lim_{m \to \infty} \frac{1}{(m-1)!} \int_{v_1 < v_2 < \ldots < v_m} g(v_1) \ldots g(v_m) \lambda^m e^{-\lambda v_m} dv_1 \ldots dv_m
\]

\[
= \lim_{m \to \infty} \frac{1}{(m-1)!} \int_{0<v_1<v_m} \int_{0<v_2<v_m} \ldots \int_{0<v_m} g(v_1) \ldots g(v_m) \lambda^m e^{-\lambda v_m} dv_1 \ldots dv_m \quad (15)
\]

the number of equal integrals is given by the coefficient \( (m-1)! \), which represents the total number of permutations of first \( m-1 \) variables. The Jacobian of the transformation is equal to 1. It follows:

\[
\mathbb{E}(u^N_t) = \lim_{m \to \infty} \frac{1}{(m-1)!} \int_0^\infty \left( \int_0^\alpha g(v) dv \right)^{m-1} g(\alpha) \lambda^m e^{-\lambda \alpha} d\alpha
\]

\[
= \lim_{m \to \infty} \int_0^\infty \frac{1}{m!} \left( \lambda \int_0^\alpha g(v) dv \right)^m \lambda e^{-\lambda \alpha} d\alpha
\]

\[
= \lim_{m \to \infty} \int_0^\infty \frac{1}{m!} \left( \lambda \int_0^{u^\mathbb{I}_{[0,\xi]}(\alpha)} dv \right)^m \lambda e^{-\lambda \alpha} d\alpha
\]

\[
= \lim_{m \to \infty} \int_0^\infty \frac{1}{m!} (\lambda (\alpha + t(u-1)))^m \lambda e^{-\lambda \alpha} d\alpha
\]

\[
= \lim_{m \to \infty} \left( 1 + \ldots + \frac{1}{m!} (\lambda t(u-1))^m \right)
\]

\[
= e^{\lambda t(u-1)}. \quad \square
\]
From Fig. 2-5 in Appendix A, we note that a higher level of $\lambda$ produces more frequent jumps. Furthermore note that a $\lambda < 1$ means a longer time to the next jump. The process, $(N_t)_{t \geq 0}$, defined in Theorem 3.1 counts the number of events in the interval $[0, t]$ and from Theorem 2.7 it is a Markov jump process. The construction of the process through the specification of the inter-arrival distribution is susceptible of generalisations towards the notion of renewal processes (see Grandell (1991)).

From Theorem 3.1 we note that the time at which the events (arrivals) occur, is a random variable. In particular the inter-arrival time, $T$, follows an exponential distribution. For this distribution the memoryless property holds:

$$P(\{T > s + t\} | \{T > t\}) = P(\{T > s\}),$$

or equivalently, by Bayes theorem:

$$P(\{T > s + t\} | \{T > t\}) = \frac{P(\{T > s + t\} \cap \{T > t\})}{P(\{T > t\})} = \frac{P(\{T > s + t\})}{P(\{T > t\})},$$

$$\Leftrightarrow P(\{T > s + t\}) = P(\{T > s\}) P(\{T > t\})$$

(16)

Another distribution strictly related to the Poisson probability model is the waiting time distribution. Denote the waiting time for the $i$-th event, with:

$$\tau_i = \xi_1 + \ldots + \xi_i.$$  

(17)

Its cumulative density function is:

$$F_{\tau_n} = P(\{\tau_n \leq t\}) = 1 - P(\{\tau_n > t\})$$

$$= 1 - P(N_t \leq n - 1) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$  

(18)

By differentiating the c.d.f. with respect to $t$ we obtain:

$$f_{\tau_n} = \frac{d}{dt} \left[ 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \right]$$

$$= \lambda e^{-\lambda t} - \sum_{k=1}^{n-1} \frac{k \lambda (\lambda t)^{k-1} e^{-\lambda t}}{k!} + \sum_{k=1}^{n-1} \frac{\lambda (\lambda t)^k e^{-\lambda t}}{k!}$$

$$= \lambda e^{-\lambda t} \left[ 1 - \sum_{k=1}^{n-1} \frac{(\lambda t)^{k-1}}{(k-1)!} + \sum_{k=1}^{n-1} \frac{(\lambda t)^k}{k!} \right]$$

$$= \lambda e^{-\lambda t} \left[ 1 - \sum_{k=1}^{n-1} \frac{(\lambda t)^{k-1}}{(k-1)!} + \sum_{k=0}^{n-2} \frac{(\lambda t)^k}{k!} \right]$$

$$= \lambda e^{-\lambda t} \left[ 1 - 1 + \frac{(\lambda t)^{n-1}}{(n-1)!} \right] = e^{-\lambda t} \lambda^n t^{n-1}$$

(19)
which is the gamma distribution: $Ga(n, \lambda)$, also known in risk theory as

Erlang distribution.

The following useful properties hold for a Poisson process.

**Theorem 3.2.** Let $\mathcal{F}_t = \sigma \{ N_s, s \leq t \}$ the $\sigma$-field generated by the Poisson process $(N_t)_t$, then the processes:

(a) $\tilde{N}_t = N_t - \lambda t$ (compensated Poisson process)

(b) $M_t = (N_t - \lambda t)^2 - \lambda t = \tilde{N}_t^2 - \lambda t$

(c) $A_t = e^{a(N_t - \lambda(t^2 - 1))}$ (exponential martingale process) with $a \in \mathbb{R}$; are $\mathcal{F}_t$-martingales.

**Proof.**

(a) Note that the intensity of the process plays the role of compensator of the Poisson process and this is a general result in point process theory. We need to prove that $\mathbb{E} \left( \tilde{N}_t - \tilde{N}_s \mid \mathcal{F}_s \right) = 0$, with $t \geq s$, or equivalently: $\mathbb{E} (N_t - N_s \mid \mathcal{F}_s) - \lambda (t - s) = 0$. From the Definition 3.2 of Poisson process:

$$\mathbb{P} (N_t - N_s \mid \mathcal{F}_s) = \mathbb{P} (N_t - N_s) = \mathcal{Poi} (\lambda (t - s)),$$

thus $\mathbb{E}_P (N_t - N_s \mid \mathcal{F}_s) = \mathbb{E}_P (N_t - N_s) = \lambda (t - s)$.

(b) We need to prove that:

$$\mathbb{E} (M_t - M_s \mid \mathcal{F}_s) = \mathbb{E} \left( (N_t - \lambda t)^2 - (N_s - \lambda s)^2 - \lambda (t - s) \mid \mathcal{F}_s \right) = 0.$$

From the Definition 3.2 of Poisson process it follows that:

$$\mathbb{P} (N_t - N_s \mid \mathcal{F}_s) = \mathbb{P} (N_t - N_s) = \mathcal{Poi} (\lambda (t - s)),$$

thus:

$$\mathbb{E}_P \left( (N_t - N_s - \lambda (t - s))^2 \mid \mathcal{F}_s \right) = \mathbb{E}_P \left( (N_t - N_s - \lambda (t - s))^2 \right) =$$

$$\text{Var} (N_t - N_s) = \lambda (t - s).$$

Now it is sufficient to observe that:

$$\mathbb{E}_P \left( (N_t - N_s - \lambda (t - s))^2 \mid \mathcal{F}_s \right) =$$

$$= \mathbb{E}_P \left( (N_t - \lambda t)^2 + (N_s - \lambda s)^2 - 2 (N_t - \lambda t) (N_s - \lambda s) \mid \mathcal{F}_s \right)$$

$$= \mathbb{E}_P \left( (N_t - \lambda t)^2 + (N_s - \lambda s)^2 \mid \mathcal{F}_s \right) - 2 \mathbb{E}_P ((N_t - \lambda t) (N_s - \lambda s) \mid \mathcal{F}_s)$$

and because $(N_t - N_t)$ is a $\mathcal{F}_t$-martingale and $(N_s - \lambda s)$ is $\mathcal{F}_s$-measurable,

$$\mathbb{E}_P \left( (N_t - \lambda t)^2 + (N_s - \lambda s)^2 \mid \mathcal{F}_s \right) - 2 \mathbb{E}_P ((N_s - \lambda s) (N_s - \lambda s) \mid \mathcal{F}_s) =$$

$$= \mathbb{E}_P \left( (N_t - \lambda t)^2 - (N_s - \lambda s)^2 \mid \mathcal{F}_s \right) = \mathbb{E}_P \left( \tilde{N}_t - \tilde{N}_s \mid \mathcal{F}_s \right)$$
(c) The proof is similar to previous points. Here it is sufficient to note that
\[ \mathbb{E}_F (e^{aN_{t-s} N_t} \mid \mathcal{F}_s) = e^{\lambda (t-s)(e^s-1)}. \]

\[ \Box \]

Just to give a graphical representation of these process, in the Appendix B we simulate a path from each of the previous martingales, associated to a Poisson process with intensity \( \lambda = 0.5 \). For an introduction to the simulation method for the Poisson process we refer the interested reader to Reply (1987). Note however that the following property is particularly useful, because it shows an easy way to simulate a path from a Poisson process.

**Theorem 3.3.** - Let \( 0 = t_0 < t_1 < \ldots < t_k = t \) be a partition of the interval \([0,t]\) and \( (N_t)_{t \geq 0} \) a Poisson process, then conditionally to \( N_t = n \) the random variables \( N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, \ldots, N_{t_k} - N_{t_{k-1}}, N_{t_1} \) are independent, uniformly distributed in the interval \([0,t]\).

The following differential derivation of the Poisson model (see Ross (1997)), could help to understand other simulation methods, which have been proposed in the literature.

The process \( N_t \) describes the number of events in a time interval \([0,t]\). Assume the following hypotheses hold:

(a) \( N_{t=0} = 0; \)

(b) \( \forall t_1 \leq t_2 < t_3 \leq t_4 \) the number of events that occur in disjoint intervals \([t_1,t_2]\) and \([t_3,t_4]\) are independent;

(c) the probability distribution of the number of events \( N_t \), that occur in the interval \([t,s]\), depends only on the length of the interval, \( (s-t) \), and not on the time location;

(d) assume a time interval of length \( dt \), then \( \mathbb{P} (N_{t,t+dt} = 1) = \lambda dt + o(dt) \) and \( \mathbb{P} (N_{t,t+dt} \geq 2) = o(dt) \).

Consider a partition of the time interval \([0,t]\) in \( n \) in disjoint subintervals of length \( t/n \). There are two possible ways of obtaining \( k \) events in the interval:

(a) \( k \) intervals contain exactly one event and the other \( n-k \) intervals do not contain events;

(b) there exists at least one interval that contains two or more events.
In case (b) the probability is:

\[
\mathbb{P}(\{k \text{ events in } [0, t]\} \cap \{\text{at least 1 interval contain 2 or more events}\}) \\
\leq \mathbb{P}(\{\text{at least 1 interval contains 2 or more events}\}) \\
= \mathbb{P}\left(\bigcup_{j=1}^{n} \{j\text{-th interval contains 2 or more events}\}\right) \\
\leq \sum_{j=1}^{n} \mathbb{P}(\{j\text{-th interval contains 2 or more events}\}) = \sum_{j=1}^{n} o\left(\frac{t}{n}\right) \\
= n \frac{t}{n} o\left(\frac{t}{n}\right) = o\left(\frac{t}{n}\right) \xrightarrow{n \to \infty} 0
\]

because when \(n \to \infty\), \(t/n \to \infty\) and \(o(t/n)/t/n \to 0\).

In case (a) the probability of \(k\) intervals with one events and \(n-k\) intervals with zero events is given by a binomial probability model. By assumption the \(P(1 \text{ event}) = \frac{t}{n} + o\left(\frac{t}{n}\right)\) while \(P(0 \text{ events}) = 1 - \mathbb{P}(1 \text{ event or more}) = 1 - \lambda t/n - o\left(\frac{t}{n}\right) - o\left(\frac{t}{n}\right)\). Thus the probability is:

\[
\mathbb{P}(k \text{ intervals with 1 event}, n-k \text{ intervals with 0 events}) = \\
\binom{n}{k} \left(\frac{t}{n} + o\left(\frac{t}{n}\right)\right)^k \left(1 - \lambda t/n - o\left(\frac{t}{n}\right)\right)^{n-k}
\]

(20)

When \(n \to \infty\), the limiting probability is:

\[
\mathbb{P}(N_t = k) = \lim_{n \to \infty} \mathbb{P}(k \text{ intervals with 1 event}, n-k \text{ intervals with 0 events}) = \\
\binom{n}{k} \left(\lambda t/n + o\left(\frac{t}{n}\right)\right)^k \left(1 - \frac{t}{n} \left(\lambda - o\left(\frac{t}{n}\right)\right)\right)^{n-k} = \frac{1}{k!} (\lambda t)^k e^{-\lambda t}
\]

(21)

because:

\[
\lim_{n \to \infty} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = \lim_{n \to \infty} \frac{1}{k!} \left(\frac{n}{n}\right)^k \left(\frac{n-1}{n}\right) \ldots \left(\frac{n-k}{n}\right) = \frac{1}{k!}
\]

(22)

\[
\lim_{n \to \infty} \left(\lambda t + o\left(\frac{t}{n}\right)\right)^k = (\lambda t)^k
\]

(23)

\[
\lim_{n \to \infty} \left(1 - \frac{t}{n} \left(\lambda + o\left(\frac{t}{n}\right)\right)\right)^n = \lim_{x \to \infty} \left(1 + \frac{1}{x} + o\left(\frac{t}{n}\right)\right)^{-\lambda t} = e^{-\lambda t}
\]

(24)

\[
\lim_{n \to \infty} \left(1 - \frac{t}{n} \left(\lambda - o\left(\frac{t}{n}\right)\right)\right)^{-k} = 1.
\]

(25)
We conclude this section with the definition of Laplace transform of a Poisson process. It is useful to characterize the Poisson process through its Laplace transform, in order to analyse some powerful generalisations, like the Cox and the compound Poisson processes (see Section 4), which are used in credit risk modelling.

**Theorem 3.4.** (Laplace transform) - Let \((N_t)_{t \geq 0}\) be a Poisson process with intensity \(\lambda\), then its Laplace transform is:

\[
\Psi_N = E\left(e^{-N_t}\right) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} \lambda^k}{k!} e^{-\lambda t} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(e^{-\lambda t})^k}{k!} = e^{-\lambda t (1-e^{-1})}.
\]

(26)

4 Generalizations of the Poisson Process

Before proposing some generalisations of the homogeneous Poisson process based on a redefinition of the underlying probability model, we introduce the operations of superposition and composition of Poisson processes and characterize the result of such operations. The operation of superposition is useful in order to decompose the intensity function in more simple functions before using simulation techniques. The operation of composition, through random sums, is a fundamental tool used in risk theory and also in the actuarial approach to credit risk modelling (see Crédit Suisse (1997) credit portfolio approach), because it allows to aggregate through a random sum a set of competing risk sources.

**Theorem 4.1.** (Superposition) - Let \(N_{1t}, N_{2t}, \ldots, N_{nt}\) be a sequence of independent Poisson processes, with intensities \(\lambda_1, \lambda_2, \ldots, \lambda_n\) respectively. The superposition of these processes:

\[
M_t = N_{1t} + N_{2t} + \ldots + N_{nt}
\]

(27)
gives a Poisson process with intensity: \(\lambda = \lambda_1 + \ldots + \lambda_n\).

**Proof.** Follows immediately by calculating the Laplace transform of the process \(M_t\).

\(\Box\)

**Definition 4.1.** (Compound Poisson process) - Let \(\xi_1, \xi_2, \xi_3, \ldots\) be a sequence of independent and identically distributed random variables with c.d.f. \(F_\xi\) and Laplace transform \(E\left(e^{u\xi}\right) = g(u)\). Let \(N_t\) be a Poisson process with intensity \(\lambda\) independent of the sequence \(\xi_1, \xi_2, \xi_3, \ldots\). The process:

\[
M_t = \sum_{k=1}^{N_t} \xi_k
\]

(28)
is called compound Poisson process.

**Theorem 4.2.** - The compound Poisson process $M_t$ has Laplace transform $e^{-\lambda(1-g(u))}$.

**Proof.**

\[
E(e^{M_t u}) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} \mathbb{E} \left( \exp \left( u \sum_{k=1}^{N_t} \xi_k \right) \mid N_t = j \right) \\
= \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} \mathbb{E} \left( \exp \left( u \sum_{k=1}^{j} \xi_k \right) \right) \\
= \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} \prod_{k=1}^{j} \mathbb{E} \left( e^{u \xi_k} \right) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} (g(u))^j \\
= \sum_{j=0}^{\infty} \frac{(\lambda t g(u))^j}{j!} e^{-\lambda t} = e^{-\lambda t(1-g(u))} \\
\tag{29}
\]

From Definition 3.2 it is possible to build other definitions of Poisson process through a modification of the hypothesis on the intensity parameter. A first simple extension follows by assuming the intensity of the arrivals is a deterministic function of the time.

**Definition 4.2.** *(Deterministic intensity)* - Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(N_t)_{t \geq 0}$ a stochastic process. Let $G_t = \sigma \{ N_s, s \leq t \}$ the $\sigma$-field generated by the process, then $(N_t)_{t \geq 0}$ is called in-homogeneous Poisson process with intensity function (or hazard rate function) $\lambda(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ if the following two conditions are satisfied:

1. $\forall 0 \leq s < t$ the increment $(N_t - N_s)$ is independent of the $\sigma$-field $G_s$;
2. $\forall 0 \leq s < t$ the increment $(N_t - N_s)$ has the following Poisson distribution:

\[
\mathbb{P} \left\{ (N_t - N_s) = k \mid \mathcal{F}_s \right\} = \mathbb{P} \left\{ (N_t - N_s) = k \right\} = \left( \frac{\Lambda(t) - \Lambda(s)}{k!} \right)^k e^{-(\Lambda(t) - \Lambda(s))} \\
\text{with } k = 0, 1, 2, \ldots
\]
The function: $\Lambda(t) = \int_0^t \lambda(u) \, du$ is called intensity measure (or hazard function, or cumulative intensity) (see for example Cox and Oakes (1984), Zacks (1992), Ibrahim, Chen and Sinha (2001), for an introduction to survival models in probability and statistics). Note that the increments of an inhomogeneous Poisson process are no longer stationary unless the intensity $\Lambda(t)$ is a linear function of $t$. In the following, we give two simple examples of time-dependent deterministic intensity function.

**Example 1. (Piecewise constant intensity) -** Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of the time interval $[0,T]$, it is possible to construct an in-homogeneous Poisson process simply by assuming that the intensity is a piecewise constant function:

$$\lambda(t) = \sum_{i=0}^{n-1} \lambda_i 1_{\{t_i < t \leq t_{i+1}\}}.$$

In Figure 9, Appendix C, we show a simulated path from a process with a piecewise constant intensity. The time interval is divided in two sub-intervals, notice from the simulated path, that the intensity of the jump is higher in the second sub-interval because an higher intensity parameter is assumed.

**Example 2. (Weibull intensity) -** A widely used intensity function is the power function with two parameters: $\lambda(t) = \alpha \beta t^{\beta-1}$.

This kind of intensity characterizes a renewal process with a Weibull inter-arrival time distribution (see Section 1). In Fig. 11 and 12, we simulate some paths from this process through a change of time technique and under different assumptions on the parameters. In particular notice that if $\beta < 1$ then the probability of the jump events decreases with time, while if $\beta > 1$ then the probability increases with time. The intermediate case $\beta = 1$ corresponds to a constant intensity function. The same considerations follow from Fig.1, which shows the value of the cumulated intensity function for different values of the parameters.

The change of time is a general method for building an in-homogeneous Poisson process. This method is also very useful in order to simulate from an in-homogeneous Poisson process. Let $(\tilde{N}_t)_{t \geq 0}$ be a standard Poisson process, with constant intensity, $\lambda = 1$, defined on the filtered probability space $(\Omega, \mathcal{F}, \tilde{G}, \mathbb{P})$, where $\tilde{G} = (\tilde{G}_t)_{t \geq 0}$ and $\tilde{G}_t = \sigma \{ \tilde{N}_s, s \leq t \}$. Introduce an operational time scale function, given by the cumulated intensity, $\Lambda(t)$, and assume it is a continuous function (generally obtained by cumulating over time the intensity rate function as in Definition 4.2). The process defined
Figure 1: Power change of time for three Weibull intensity processes, with:
$\alpha = 0.5, \beta = 0.5$, $\alpha = 0.5, \beta = 1$ and $\alpha = 0.5, \beta = 1.5$.

by the following *time-change* operation: $N_t = \tilde{N}_t \circ \Lambda (t) \overset{\text{def}}{=} \tilde{N}_{\Lambda(t)}$ is an in-homogeneous Poisson process with intensity $\Lambda (t)$. The proof follows from Definition 3.2 of Poisson process, where the increments are defined as:

$$
\mathbb{P}\{ (\tilde{N}_{\Lambda(t)}) - \tilde{N}_{\Lambda(s)} = k \} = \frac{(\Lambda (t) - \Lambda (s))^k}{k!} e^{-\Lambda(t)-\Lambda(s)}
$$

which is equivalent to the definition of in-homogeneous Poisson process with intensity $\Lambda (t)$. The homogeneous Poisson process is stationary because it does not allow for fluctuations in the mean number of events, but in order to produce a more realistic stochastic model it is convenient to modify this hypothesis. The introduction of a deterministic time-dependent intensity is a first generalization, but it also possible to introduce random fluctuations in the intensity function. Assume the hazard function $\Lambda(\omega)$ is a random variable with p.d.f. $f_\Lambda$, then through the usual time-change operation we obtain the following process.

**Definition 4.3.** *(Mixed Poisson process)* - Let $(\tilde{N}_t)_{t \geq 0}$ be a standard homogeneous Poisson process with intensity $\lambda = 1$ and $\Lambda(\omega)$ a random variable with $\mathbb{P}\{ \Lambda > 0 \} = 1$, independent of $(\tilde{N}_t)_{t \geq 0}$. Then the process $N_t = \tilde{N}_t \circ (\Lambda(\omega) t) \overset{\text{def}}{=} \tilde{N}_{\Lambda(\omega)t}$ is called a mixed Poisson process. The random variable $\Lambda(\omega)$ is called structure variable.
Notice that a mixed Poisson process has stationary increments, however the independent increments condition does not hold, unless the distribution of \( \Lambda \) is concentrated at one point: \( f_\Lambda = \delta_{\Lambda^*} \). Assume \( 0 \leq t_1 < t_2 < t_3 < t_4 \leq t \) then the covariance between the increments is:

\[
E((N_{t_2} - N_{t_1})(N_{t_4} - N_{t_3})) - E(N_{t_2} - N_{t_1})E(N_{t_4} - N_{t_3}) = \\
E[E((N_{t_2} - N_{t_1})(N_{t_4} - N_{t_3}) \mid \Lambda)] + \\
- E[E(N_{t_2} - N_{t_1} \mid \Lambda)]E[E(N_{t_4} - N_{t_3} \mid \Lambda)] = \\
E(\Lambda (t_2 - t_1) \Lambda (t_4 - t_3)) - E(\Lambda (t_2 - t_1))E(\Lambda (t_4 - t_3)) = \\
Var(\Lambda) (t_2 - t_1)(t_4 - t_3).
\]

A common choice for the distribution of the structural variable \( \Lambda (\omega) \) is the gamma distribution \( \Gamma(\alpha, \beta) \) whose density function is:

\[ f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \]

with \( y \geq 0 \). Thus the resulting mixed Poisson process is a Pólya process.

**Theorem 4.3. (Pólya process)** - Let \( (N_t)_{t \geq 0} \) be a mixed Poisson process and let the structural variable \( \Lambda (\omega) \) follow a gamma distribution \( \Gamma(\alpha, \beta) \), then the process \( (N_t)_{t \geq 0} \) is a Pólya process (or negative binomial process).

**Proof.** In order to show that the resulting process is a Pólya process it is sufficient to show that the distribution of the process is a negative binomial:

\[
\mathbb{P}(N(t) = n) = \int_0^\infty \mathbb{P}(N(t) = n \mid \Lambda = \lambda) f_\Lambda(\lambda) d\lambda = \\
\int_0^\infty \mathbb{P}(\tilde{N}(t\Lambda) = n \mid \Lambda = \lambda) f_\Lambda(\lambda) d\lambda = \\
\int_0^\infty (\lambda t)^n \frac{e^{-\lambda t}}{n!} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta \lambda} \lambda^{\alpha-1} d\lambda = \\
t^n \frac{\beta^\alpha}{n!} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \frac{1}{(\beta + t)^{\alpha+n}} \int_0^\infty (\beta + t)^{\alpha+n} e^{-x(\beta+t)} x^{n-1} d\lambda = \\
\frac{(\alpha + n - 1)!}{(\alpha - 1)! n!} \left( \frac{t}{\beta + t} \right)^n \left( \frac{\beta}{\beta + t} \right)^\alpha = \\
Neg(\alpha, p) \quad \text{with} \quad p = \frac{\beta}{\beta + t}.
\]
Figure 13, in Appendix C, exhibits some paths simulated from a Polya process. We take a Gamma distributed intensity, with parameter $\alpha = 3/2$ and $\beta = 2/3$. Note that the intensity’s mean value is $\alpha \beta = 1$, but the process behaves differently with respect an homogeneous Poisson with intensity $\lambda = 1$, because the random nature of the Polya process intensity allows for a high heterogeneity of the paths. This effect can be noted by a direct comparison between Figure 13 and Figure 14 where some samples have been simulated from a homogeneous Poisson with intensity $\lambda = 1$.

The doubly stochastic Poisson process is another useful generalisation of the Poisson process. The double stochastic Poisson is also known in the literature as conditional Poisson process (see Bremaud (1981)). See Cox and Isham (1980), Kingman (1993) for further details on the doubly stochastic Poisson as point process and Grandell (1976) and (1991) for the applications of this kind of processes in risk theory. A particular case of doubly stochastic Poisson process is the Cox process (see Grandell (1991)), which is used in the reduced form approach to credit risk modelling (see Lando (1999)). The conditional Poisson process, introduced in probability theory by Cox in 1955, allows to model more realistically the random occurrence of events like the default event in credit risk model, or the number of contingent claims in the actuarial models. In this kind of process the intensity is not constant, but varies stochastically over time. In particular the intensity is assumed to be a stochastic process. It can be specified also through a dynamic model, where some state variables (business cycle variables, macro-market variables, financial indexes, etc.) determine its evolution either over time or space.

The stochastic nature of the intensity causes the variance of the process to be greater than the variance of a homogeneous Poisson process with the same expected intensity measure. This feature of the Cox processes is referred, in the literature on point processes and survival models, as overdispersion. Let us build the conditional Poisson process through the following definitions.

**Definition 4.4. (Hazard process)** - Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space and let $\mathbb{F}$ be a sub-filtration of $\mathbb{G}$. We called hazard process an $\mathcal{F}_t$-adapted, right-continuous, increasing stochastic process $(\Lambda_t)_{t \geq 0}$ with $\mathbb{P}$-a.s. $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$.

The hazard process can be specified also in the following way:

$$\Lambda_t = \int_0^t \lambda_s ds, \quad \text{with} \quad t \geq 0$$

where $(\lambda_t)_{t \geq 0}$ is a $\mathcal{F}_t$-progressively measurable process with locally Riemann integrable realizations. The process $(\lambda_t)_{t \geq 0}$ is called intensity process (or
hazard rate process). Although it is possible to define a conditional Poisson process through the usual random time-change operation, we prefer to define it by specifying the increment distribution.

**Definition 4.5. (Doubly stochastic Poisson process)** - Let \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})\) be a filtered probability space and \((N_t)_{t \geq 0}\) a stochastic process defined on it. Let \(\mathcal{F}\) be a sub-filtration of \(\mathbb{G}\), then \((N_t)_{t \geq 0}\) is called \(\mathcal{F}\)-conditional Poisson process with hazard process \((\Lambda_t)_{t \geq 0}\) if the following two conditions are satisfied:

1. \(\forall 0 \leq s < t\), conditionally \(\mathcal{F}_\infty\) on the increment \((N_t - N_s)\) is independent of the \(\sigma\)-field \(\mathcal{G}_s\);

2. \(\forall 0 \leq s < t\) the increment \((N_t - N_s)\) has the following Poisson distribution: \(\mathbb{P} \{(N_t - N_s) = k \mid \mathcal{G}_s \vee \mathcal{F}_\infty\} = \mathbb{P} \{(N_t - N_s) = k \mid \mathcal{F}_\infty\} = \frac{(\Lambda_t - \Lambda_s)^k}{k!} e^{-(\Lambda_t - \Lambda_s)}\) with \(k = 0, 1, 2, \ldots\) where \(\mathcal{F}_\infty = \sigma \{N_u : u \in \mathbb{R}_+\}\).

We adapt the definition of overdispersion, given in Kingman (1993), to our definition of Poisson process, we obtain an equality which shows that the variance of a Cox process is greater than the variance of a Poisson process with the same expected intensity measure. Let \(0 = t_0 < t_1 < t_2 < \ldots < t_n = T\) be a partition of the time interval \([0, T]\), and introduce, as in Definition 4.5, a random measure:

\[
\Lambda ([0, t]) = \Lambda_t = \int_0^t \lambda(u) \, du.
\]

We need to prove that \(\text{Var} (N_t) \geq \mathbb{E} (N_t)\). Observe that \(\mathbb{E} (N_t) = \mathbb{E}_\Lambda (\mathbb{E}_P (N_t \mid \Lambda_t)) = \mathbb{E}_\Lambda (\Lambda_t)\) and compute:

\[
\begin{align*}
\mathbb{E} \left( (N_t)^2 \right) &= \mathbb{E}_\Lambda \left( \mathbb{E}_P \left( (N_t)^2 \mid \Lambda_t \right) \right) \\
&= \mathbb{E}_\Lambda \left( \Lambda_t + \Lambda_t^2 \right) = \mathbb{E}_\Lambda (\Lambda_t) + \text{Var}_\Lambda (\Lambda_t) + (\mathbb{E}_\Lambda (\Lambda_t))^2 \\
&\iff \mathbb{E} \left( (N_t)^2 \right) - (\mathbb{E}_\Lambda (\Lambda_t))^2 = \mathbb{E}_\Lambda (\Lambda_t) + \text{Var}_\Lambda (\Lambda_t) \\
&\iff \text{Var} (N_t) = \mathbb{E} (N_t) + \text{Var}_\Lambda (\Lambda_t) .
\end{align*}
\]

The desired result follows immediately.

The hazard process plays an important role in the martingale approach to credit risk because it is the compensator of the associated doubly stochastic Poisson process (see for example Bielecki and Rutkowski (2002)). The following theorem states the result.
Theorem 4.4. - Let $(N_t)_{t \geq 0}$ be a doubly stochastic Poisson process with intensity $(\Lambda_t)_{t \geq 0}$, then the compensated process $\tilde{N}_t = N_t - \Lambda(t)$ is a $\mathcal{G}$-martingale.

Proof. We need to prove that $E_P(\tilde{N}_t - \tilde{N}_s | \mathcal{F}_s) = 0$ with $t \geq s$, or equivalently $E(N_t - N_s - \Lambda_t | \mathcal{G}_s) = 0$. From the Definition 4.5 of doubly stochastic Poisson process: $P(N_t - N_s | \mathcal{G}_s) = P(N_t - N_s | \mathcal{F}_s)$. Thus

$$E_P(N_t - N_s | \mathcal{G}_s) - E_P(\Lambda_t - \Lambda_s | \mathcal{G}_s) = E_P(N_t - N_s | \mathcal{G}_s) - E_P(N_t - N_s | \mathcal{F}_s) = E_P(\Lambda_t - \Lambda_s | \mathcal{G}_s) - E_P(\Lambda_t - \Lambda_s | \mathcal{F}_s) = 0.$$

In some applications like the credit risk models it is useful to consider a particular kind of conditional Poisson process which is called Cox process. In this kind of process the intensity process, $\lambda_t = h(t; X_t)$, is a continuous function of the time and of a vector of state variables $X_t$, which are $\mathbb{R}^d$-valued stochastic processes independent of the Poisson process. A simple way of constructing the filtrations related to the Cox process is the following: $\mathcal{H}_t = \sigma \{ N_s : s \leq t \}$, $\mathcal{F}_t = \sigma \{ X_s : s \leq t \}$ and finally $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$. We present some examples of intensity process and refer to Bielecki and Rutkowski (2002) for a more complete analysis of the hazard function (or process) modelling.

Example 3. (Intensity diffusion process) - In the continuous time model of Schönbucher (2000) the intensity process $\lambda_t$ is the solution of the following SDE:

$$d\lambda_t = \lambda_t \mu dt + \lambda_t \sigma dW_t.$$

Example 4. (Proportional hazard rate process) - As in the regression model due to Cox (1972), the hazard rate process depends generally on both time and state variables: $\lambda_t = h(t, X_t)$. When the process can be factorised as follows: $\lambda_t = h(t, X_t) = h_0(t) \exp(g(X_t))$ the model is called proportional hazard. The function $h_0(t)$ is called baseline hazard function and the function $g(x)$ is generally assumed to be linear: $g(x) = \beta^T x$, with $\beta$ vector of coefficients.

Given a Cox process it is possible to determine the density function of the associated random time.

Theorem 4.5. - Let $(\Omega, \mathcal{G}, \mathcal{G}, \mathbb{P})$ be a filtered probability space with $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ and $(X_t)_{t \geq 0}$ a $\mathbb{R}^d$-valued continuous stochastic process. Let $\lambda(X_t) : \mathbb{R}^d \to \mathbb{R}_+$ be the intensity process and $E$ an exponential random variable,
independent of $X$, with p.d.f.: $P(E \geq t) = e^{-t}$. The random time defined as:

$$\tau = \inf \left\{ t \geq 0 : \int_0^t \lambda (X_s) \, ds \geq E \right\}$$

(33)

has the following conditional distribution function given the $\sigma$-field $\mathcal{F}_t$:

$$P(\{\tau > s\} \mid \mathcal{F}_t) = \exp \left( - \int_0^s \lambda (X_u) \, du \right)$$

(34)

The unconditional distribution is given by:

$$P(\{\tau > s\}) = \mathbb{E} \left( \exp \left( - \int_0^s \lambda (X_u) \, du \right) \right).$$

(35)

5 Point processes

The Poisson process can intuitively described as a sequence of unitary jumps, which occur at random on a time interval. Thus the process can be viewed as a random partition of points on the set $\mathbb{R}_+$. In this paragraph we generalize the Poisson process by modifying the assumption on the randomly partitioned set and on the probability function associated to the subsets. The resulting processes are called point processes. Let $(E, \mathcal{E})$ be a space with the borel $\sigma$-algebra, $z_1, z_2, z_3, ..., z_i, ...$ a sequence of point and $B$ a subset of $E$. Then the point measure of the set $B$ is given by:

$$m(B) = \sum_{i=1}^{\infty} I_{\{z_i \in B\}}$$

Definition 5.1. (Point process) - Let $M_p = \{ m(B) : E \rightarrow \mathbb{N} \}$ be the set of all the finite point measures defined on $(E, \mathcal{E})$ and let $\mathcal{M}_p = \sigma \{ m : m \rightarrow m(A), A \in \mathcal{E} \}$. A point process is a random variable defined as:

$$N(\omega) : (\Omega, \mathcal{F}, P) \rightarrow (M_p, \mathcal{M}_p)$$

(36)

Note that the probability $P_N$, defined on $(M_p, \mathcal{M}_p)$, is the image of the probability $P$, through the process $N$. The probability, $P_N$, is determined by the joint probability of the random vector $(N(\omega, B_1), N(\omega, B_2), ..., N(\omega, B_k))$. 

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where \( B_1, B_2, \ldots, B_k \in \mathcal{E} \). If the \( N \) is a point process, the \textit{concentration measure} (or \textit{intensity}) \( \mu \) is defined as: \( \mu (A) = \mathbb{E} (N (\omega, A)) \), \( \forall A \in \mathcal{E} \). It is also possible to construct a Poisson point process by specifying the concentration measure of the process. The following proposition states the result.

**Theorem 5.1.** (Point Poisson process) - Let \( \mu \) be a \( \sigma \)-finite positive measure defined on \((E, \mathcal{E})\), then it exists a point process \( N \) with values in \( M_p (E) \) such that:

1. \( \forall A \in \mathcal{E} \) the random variable \( N (\omega, A) \) follows a Poisson distribution with intensity \( \mu (A) \)

2. \( A_1, A_2, \ldots, A_k \in \mathcal{E} \) such that \( A_i \cap A_j = \emptyset \ \forall i \neq j \) the random variables \( N (\omega, A_1), N (\omega, A_2), \ldots, N (\omega, A_k) \) are independent.

We report the proof due to Bouleau (2000) of previous theorem because it shows the importance of the Laplace transform in the study of the point processes.

**Proof.** Assume the measure \( \mu \) is finite: \( \mu (E) = \theta < \infty \), then by normalizing the measure we obtain a probability: \( \mu_0 = \mu / \theta \). Assume a point process defined on the following space: \((\Omega, \mathcal{F}, \mathbb{P}) = (N, \mathcal{P} (N), \mathbb{P} (\theta)) \times (E, \mathcal{E}, \mu_0)^N\), where \( \mathbb{P} (\theta) \) represents the Poisson distribution with parameter \( \theta \). The point process is defined as follows:

\[
N (\omega) = \sum_{i=1}^{Y(\omega)} \delta_{X_i} (\omega)
\]

with \( Y, X_1, X_2, \ldots, X_i, \ldots \in (N, \mathcal{P} (N), \mathbb{P} (\theta)) \times (E, \mathcal{E}, \mu_0)^N\)

To show that \( N \) is a Poisson process it is sufficient to calculate its Laplace transform:

\[
\mathbb{E} \left( u^{N(\omega, A)} \right) = \mathbb{E} \left( \frac{Y(\omega)}{u} \sum_{i=1}^{Y(\omega)} 1_{A_1} (X_i) \right) = \mathbb{E} \left( \prod_{i=1}^{Y(\omega)} u^{1_{A_1} (X_i)} \right)
\]

\[
= \mathbb{E} \left( \sum_{n=0}^{\infty} \mathbb{P} (\{Y = n\}) \prod_{i=1}^{n} u^{1_{A_1} (X_i)} \right) = \sum_{n=0}^{\infty} e^{-\theta} \frac{\theta^n}{n!} \prod_{i=1}^{n} \mathbb{E} \left( u^{1_{A_1} (X_i)} \right)
\]

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\[= \sum_{n=0}^{\infty} \frac{e^{-\theta} \theta^n}{n!} (1 \cdot \mathbb{P}(\{X_i \notin A\}) + u \cdot \mathbb{P}(\{X_i \in A\}))^n\]
\[= \sum_{n=0}^{\infty} \frac{e^{-\theta} \theta^n}{n!} (1 \cdot (1 - \mu_0(A)) + u \cdot \mu_0(A))^n\]
\[= \sum_{n=0}^{\infty} \frac{e^{-\theta} \theta^n}{n!} (\theta (1 + (u - 1) \mu_0(A)))^n = e^{-\theta + \theta(1+(u-1)\mu_0(A))}\]
\[= e^{\theta(u-1)\mu(A)} = e^{(u-1)\mu(A)}\]

which is the Laplace transform of a Poisson process with intensity \(\mu(A)\). Furthermore if the \(A_1, A_2, \ldots, A_k \in \mathcal{E}\) are independent then:

\[\mathbb{E}\left(\frac{N^{N(\omega,A_1)} u_1^{N(\omega,A_2)} \cdots u_k^{N(\omega,A_k)}}{\mu(A)\mu(A_2)\cdots\mu(A_k)}\right) = e^{(u_1-1)\mu(A_1)} e^{(u_2-1)\mu(A_2)} \cdots e^{(u_k-1)\mu(A_k)}\]  \hspace{1cm} (37)

for the properties of the Laplace transform also the random variables \(N(\omega, A_1), N(\omega, A_2), \ldots, N(\omega, A_k)\) are independent. \(\square\)

We conclude this section by giving the Laplace transform of a point Poisson process. It is a generalization of the result stated in Theorem 3.4.

**Theorem 5.2. (Laplace transform)** - The Laplace transform of a point Poisson process with concentration measure \(\mu\) is given by:

\[\Psi_N = \exp \left\{ - \int \left(1 - e^{-f(x)}\right) d\mu(x) \right\}.\]  \hspace{1cm} (38)

### 6 Conclusion

Many processes are now involved in credit risk modelling. Much attention in the literature has been devoted to diffusion processes, while other kind of processes, like jump processes might be less known. Jump processes play an important role in both defaultable claim pricing and credit portfolio loss evaluation, because they are suitable to capture the discontinuity induced by a default or credit migration event. In this work we focus on the homogeneous and in-homogeneous Poisson processes, highlighting main features and differences. Some analytical and simulated simple examples help the reader to understand the behaviour of the processes analysed.
Appendix A - The Intensity Parameter

In this appendix the present some paths simulated from a Poisson process, with the following intensities: $\lambda = 0.05, 0.5, 1, 1.5$. Observe how higher is the value of the intensity parameter, more frequent are the jumps.

![Figure 2: Simulated path from a Poisson process with $\lambda = 0.05$.](image)

![Figure 3: Simulated path from a Poisson process with $\lambda = 0.5$.](image)
Figure 4: Simulated path from a Poisson process with $\lambda = 1.00$.

Figure 5: Simulated path from a Poisson process with $\lambda = 1.50$. 
Appendix B - Martingales associated to a Poisson process

In the following we exhibits some paths simulated from the martingale processes defined in Theorem 3.2:

a) $\tilde{N}_t = N_t - \lambda t$ (compensated Poisson process);
b) $M_t = N_t^2 - \lambda t$;
c) $A_t = e^{a\tilde{N}_t - \lambda (e^a - 1)}$ (exponential martingale process).

The simulated martingale processes are associated to the Poisson process given in Fig. 4 (with $\lambda = 0.5$). The exponential martingale is represented (Fig. 8) in logarithmic scale.

Figure 6: A path simulated from the compensated Poisson process, $\tilde{N}_t = N_t - \lambda t$. 
Figure 7: A path simulated from the martingale $M_t = \tilde{N}_t^2 - \lambda t$.

Figure 8: A path simulated from the exponential martingale process, $A_t = e^{\alpha \tilde{N}_t - \lambda (e^\alpha - 1)}$. 
Appendix C - In-homogeneous Poisson Process

In the following we show some simulated examples of the in-homogeneous Poisson process. We carried out the simulation by means of a change of time technique. In the first example (Fig. 9) we assume that the Poisson process has the following piecewise constant intensity function:

\[
\lambda(t) = \begin{cases} 
\lambda_1 & \text{if } 0 \leq t \leq 100 \\
\lambda_2 & \text{if } 100 < t \leq 150 
\end{cases} \quad \text{with} \quad \lambda_1 = 0.05, \lambda_2 = 1.5 \quad (39)
\]

Observe that the probability of a jump of the process is higher in the second time interval (see Figure 9). In the second example (Figures 10 and 11) we still assume a time dependent deterministic intensity function. In particular the intensity is the same of a renewal process with a Weibull inter-arrival time (duration time) distribution:

\[
\lambda(t) = \alpha \beta t^{\beta-1} 
\]

If you compare the simulated path of a standard Poisson process (Fig. 10) with the path of the in-homogeneous processes (Fig. 11 and 12), you can see the effect of the parameter beta on the random times stream. When beta is greater then 1, the probability of a jump increases with time (Fig. 12), when beta is less then 1 the probability decreases in t (Fig. 11). In the last example (Fig. 13) we simulate a Polya process.
Figure 9: A Poisson process with piecewise constant intensity function.

Figure 10: A Poisson process with Weibull intensity: standard Poisson sample path.
Figure 11: A Poisson process with Weibull intensity. Change of time with $\alpha = 0.4$ and $\beta < 1$.

Figure 12: A Poisson process with Weibull intensity. Change of time with $\alpha = 0.4$ and $\beta > 1$. 

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Figure 13: Samples from a Poisson process, with intensity $\lambda = 1$.

Figure 14: Mixed Poisson Process: Polya process with intensity $\Lambda \sim \mathcal{Ga}\left(\frac{3}{2}, \frac{2}{3}\right)$. 
Appendix D - Cox Process

We simulate a particular kind of Cox process, called *Markov modulated Poisson process*, which is used in some credit risk models. It is defined as:

\[ \mathbb{E}_\mathbb{P} (N_t | \mathcal{F}_t) = \Lambda_t \quad \text{with} \quad \Lambda_t = \int_0^t \alpha (X_u) \, du \quad (41) \]

with intensity function:

\[ \alpha (X_u) : \{1, \ldots, m\} \to \{\alpha_1, \ldots, \alpha_m\} \]

\[ X_u = i \mapsto \alpha (X_u = i) = \alpha_i \quad (42) \]

The process \( X_u \) is a Markov chain with values in \( 1, \ldots, m \) and with transition matrix \((Q_{i,j})_{i,j=1,\ldots,m}\). We simulate the process through the following steps:

1. simulate a sequence of random time \( \tau_1, \ldots, \tau_n \), where \( n = \sup \{i | \tau_i \leq T\} \);
2. simulate a path of the Markov chain \( X_1, \ldots, X_n \) with initial value \( X_0 \);
3. make a time-change \( \tau_i^* = \tau_i / \alpha (X_i) \).

In this numerical example we pose:

\[ Q = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}, \quad \alpha_1 = 1.7, \quad \alpha_2 = 0.2. \quad (43) \]

The values of the intensity process are exhibited in Fig. 15 while the simulated values of the Cox process are drown in Fig. 16.
Intensity process modelled as a 2-states Markov chain
Transition probabilities: \( p_{11} = 0.80, \ p_{21} = 0.20 \)

Figure 15: Two State Markov Chain.

Cox process
\( \lambda_1 = 1.70 \ (X(t) = 1), \ \lambda_2 = 0.20 \ (X(t) = 2) \)

Figure 16: Cox Process.
References


[27] Revuz D., Yor M., (1999), *Continuous Martingales and Brownian Motion*, Springer Verlag, Berlin Heidelberg.


