THE ROLE OF LONGEVITY BONDS IN OPTIMAL PORTFOLIOS

by
Francesco Menoncin

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Abstract

A longevity bond pays coupons which are proportional to the survival rate of a given population. In such a way the longevity risk becomes hedgeable on the financial market. In our model there are: (i) a longevity bond as a derivative on the population survival rate, (ii) a bond as a derivative on the stochastic instantaneously riskless interest rate, and (iii) a stock. The investor maximizes the expected (CRRA) utility of his intertemporal consumption. In such a framework we demonstrate that the amount of wealth invested in the longevity bond reduces the portfolio weight of the bond without affecting neither the weight of the stock nor the weight of the riskless asset.

JEL classification: G11.

Key words: longevity risk; dynamic programming.
1 Introduction and conclusion

In November 2004, the European Investment Bank (EIB) unveiled plans to issue the first longevity bond that offers a partial longevity risk hedge to UK pension schemes and life insurers. The EIB is rated AAA by Standard & Poor’s and Aaa1 by Moody and the new issue is part of its objective to promote economic and social cohesion within the EU.

For the longevity expertise and reinsurance capacity, the EIB relies on PartnerRe while the financial component of the longevity bond is managed by the BNP Paribas. In particular, BNP Paribas acts as structurer, manager and book-runner for the longevity bond and markets it; furthermore it enters into a swap with EIB to convert its fixed interest, longevity-linked obligations under the bond into the floating obligations, free of longevity risk, that are required by the EIB when raising funding.

The longevity bond pays a coupon given by the product between a fixed amount of money (£50,000,000 for the whole issue) and the cumulative survival rate measured on a Welsh cohort of males aged 65 in 2003. This asset has a 25 year time to maturity.

Let us show an example in Table 1 where the cumulative survival rate is computed as the product of all the previous survival rate (for instance, in 2007, 96.345% is obtained through $0.99 \times 0.988 \times 0.985$).

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mortality rate</td>
<td>1%</td>
<td>1.2%</td>
<td>1.5%</td>
</tr>
<tr>
<td>Survival rate</td>
<td>99%</td>
<td>98.8%</td>
<td>98.5%</td>
</tr>
<tr>
<td>Cumulative survival rate</td>
<td>99%</td>
<td>97.812%</td>
<td>96.345%</td>
</tr>
<tr>
<td>Coupon (on £1,000)</td>
<td>990</td>
<td>978.12</td>
<td>963.45</td>
</tr>
</tbody>
</table>

Fonte: Azzoppardi (2005)

As it can be immediately seen from Table 1, the cumulative survival rate constantly decreases and, consequently, also the longevity bond coupon decreases. Nevertheless, the coupon decreasing rate positively depends on the decreasing rate of the cumulative survival rate. This makes the longevity bond a suitable asset for hedging against the so-called longevity risk, which consists in experiencing a survival rate higher than the prospective one.

Even if the hedging role the longevity bond can play in pension fund and insurance portfolios is evident, it can be quite difficult to determine the exact amount of this bond that must be hold in optimal portfolios.

This work is aimed at determining this optimal amount of longevity bond in a framework with a perfectly competitive, arbitrage free and frictionless financial market for an economic agent who wants to maximize the expected utility of his wealth at the moment of his death. In particular, we follow the traditional route to use the stochastic dynamic programming technique (Merton, 1969,
leading to a suitable (non-linear) partial differential equation. For the method called “martingale approach” the reader is referred to Cox and Huang (1989, 1991), and Lioui and Poncet (2001).

Some closed form solutions for this problem have been found in the literature. In particular, we refer to the works of Kim and Omberg (1996), Wachter (1998), Chacko and Viceira (1999), Deelstra et al. (2000), Boulier et al. (2001), Zariphopoulou (2001) and Menoncin (2002). In all these works the market structure is as follows: (i) there exists only one stochastic state variable (the riskless interest rate or the risk premium) following the Vasicek (1977) model or the Cox et al. (1985) model, (ii) there exists only one risky asset, (iii) a bond may exist. Some works deal with a complete financial market (Wachter, 1998, Deelstra et al., 2000, and Boulier et al., 2001) while others are dealing with an incomplete market (Kim and Omberg, 1996, Chacko and Viceira, 1999, and Menoncin 2002).

In this paper we keep the market structure presented in the cited works but we take into account a stochastic terminal date. Richard (1975) already solved this kind of problem but using a deterministic law for the terminal date. Here, instead, the law which describes the behaviour of this stochastic variable is stochastic itself and so a new risk that must be hedged is introduced in the analysis. In such a framework a closed form solution to the optimal allocation problem is quite hard to find since the longevity risk is typically independent of the financial risks. The presence of a longevity bond makes this risk hedgeable.

Dahl (2004) shows how to find the price of any insurance contract when the force of mortality is stochastic. In particular, he studies the case of a force of mortality following a Cox et al. (1985) process. In this paper we do not specify any particular functional form for the force of mortality since our results are general enough to be valid for any specification of it.

Our result shows that, for an investor described by a Constant Relative Risk Aversion (CRRA) utility function, the amount of money that must optimally be invested in the longevity bond is entirely deducted from the ordinary bond without changing the weights neither of the stock nor of the riskless asset. In particular, the stock does not play any role in hedging against any risk since it just plays a speculative role.

The rest of the paper is structured as follows. Section 2 shows the model we will work on and in particular describes the financial market containing: (i) a state variable given by the stochastic riskless interest rate, (ii) a riskless asset, (iii) a bond (as a derivative on the riskless interest rate), (iv) a stock, and (v) a longevity bond (as a derivative on the survival stochastic rate). In Section 3 the optimal portfolio for the agent maximizing the expected utility of his wealth at his death time is computed and the role of the longevity bond is highlighted. In particular, we present the comparison between the optimal portfolios with and without the longevity risk. Section 4 concludes. Some technical computations are left to an appendix.
2 The model

2.1 The financial market

The instantaneously riskless interest rate \( r(t) \) is assumed to be stochastic. It follows the stochastic differential equation

\[
\begin{align*}
    dr(t) & = \mu_r(r,t) \, dt + \sigma_r(r,t) \, dW_r, \\
    r(t_0) & = r_0. 
\end{align*}
\]

One of the most common functional forms for interest rate drift and diffusion terms are

\[
\mu_r(r,t) = \alpha (\beta - r),
\]

as in Vasiček (1977) and Cox et al. (1985), and

\[
\begin{align*}
    \sigma_r(r,t) & = \sigma_r, \\
    \sigma_r(r,t) & = \sigma_r \sqrt{r},
\end{align*}
\]

as in Vasiček (1977) and Cox et al. (1985) respectively (all the parameters are assumed to be positive constant). Here, we do not specify any particular functional form.

On the financial market there are three assets:

1. a riskless asset which pays the instantaneous riskless interest rate \( r(t) \) and whose value \( G(t) \) follows

\[
\begin{align*}
    dG(t) & = G(t) \, r(t) \, dt, \\
    G(0) & = 1,
\end{align*}
\]

2. a zero coupon bond which can be thought of as a derivative on the interest rate: it is well known that in an arbitrage free market the price of any asset coincides with the expected present value of its future cash flows under the so-called martingale equivalent measure \((\mathcal{Q})\). Since the zero coupon bond pays just one monetary unit at expiration (in \( T \)), its value \( B(t,T) \) can be written as

\[
B(t,T) = \mathbb{E}_t^\mathcal{Q} \left[ e^{-\int_t^T r(s) \, ds} \right],
\]

and its dynamics is

\[
dB(t,T) = B(t,T) \, r(t) \, dt + \frac{\partial B(t,T)}{\partial r(t)} \sigma_r \, dW_r^\mathcal{Q},
\]

and so, if we call \( \xi_r \) the market price for the interest rate risk, then we can finally write

\[
\frac{dB(t,T)}{B(t,T)} = \left( r(t) + \nabla_r^B \sigma_r \xi_r \right) \, dt + \nabla_r^B \sigma_r \, dW_r,
\]

\[\text{1} \text{We recall that, under the conditions of the Girsanov Theorem, the process } dW_r^\mathcal{Q} = \xi_r \, dt + dW_r \text{ is a Wiener process.}\]
where
\[ \nabla_r^B \equiv \frac{\partial B(t,T)}{\partial r(t)} \frac{1}{B(t,T)}. \]

3. a risky asset (a stock) whose price \( S \) follows

\[
\frac{dS(t)}{S(t)} = (r(t) + \sigma_S \xi_S + \sigma_S r \xi_r) dt + \sigma_S r dW_r + \sigma_S dW_S, \tag{4}
\]

where \( \xi_S \) is the market price for the stock own risk source (the Wiener processes \( W_S \) and \( W_r \) are assumed to be independent without any loss of generality).

### 2.2 The longevity bond

A longevity bond is defined as the asset paying a coupon which is strictly proportional to the (cumulative) survival rate of a given population taken in a given moment. This is the ideal asset for hedging the longevity risk of a pension fund. In fact, while the population which subscribed to the fund increases its longevity, the fund risks to have to pay pensions for longer and longer period. Nevertheless, the increasing in longevity also means a lower decreasing rate in the longevity bond coupons. In this way, the higher pensions can be faced through the less decreasing coupons.

If we call \( p(t) \) the amount of people of a given population at time \( t \) who have survived from time \( 0 \) until \( t \) and we normalize to 1 the amount \( p(0) \), then \( p(t) \) also measures the cumulative survival rate which coincides with the survival probability.

The survival rate between \( t \) and \( s > t \) is given by

\[
\frac{p(s)}{p(t)},
\]

i.e. the number of survived people in \( s \) with respect to the number of survived people in \( t \).

In the financial literature, the survival probability \( p(t) \) is usually assumed as a deterministic function nevertheless, here, we want to take into account the case of a survival probability with a stochastic "force of mortality" \( \lambda(t) \). Since the force of mortality measures how many individuals die in each period, then we can write

\[
\frac{dp(t)}{p(t)} = -\lambda(t) dt, \quad p(0) = 1,
\]

while we assume that \( \lambda \) stochastically evolves according to

\[
d\lambda(t) = \mu_\lambda(t,\lambda) dt + \sigma_\lambda(t,\lambda) dW_\lambda, \tag{5}
\]
where we assume $W_\lambda$ to be independent of $W_r$. Here, we do not specify any particular functional form neither for $\mu_\lambda$ nor for $\sigma_\lambda$ (even if a quite natural choice could be the one already presented for the interest rate, i.e. a mean reverting process). The same structure is presented in Dahl (2004) where, in particular, the author uses an affine mortality structure (as in the interest rate specification by Cox et al. 1985).

Now, we take into account a longevity bond in which expires in $T$ and whose coupons, at any time $s$, are given by $p(s)/p(t)$ (the case of any coupon strictly proportional to the survival rate is trivial). The fundamental theorem of asset pricing tells us that the value of this bond is given by the present expected value (under a martingale equivalent measure) of all the coupons discounted at the riskless interest rate:

$$E^Q_t \left[ \int_t^T \frac{p(s)}{p(t)} e^{-\int_t^s r(u) du} ds \right].$$

In our framework, for the sake of simplicity and without losing any generality, we take into account a zero coupon longevity bond whose value $L(t,T)$ is given by

$$L(t,T) = E^Q_t \left[ \frac{p(T)}{p(t)} e^{-\int_t^T r(u) du} \right].$$

There is no loose of generality since it is evident that the longevity bond is just a linear combination of zero coupon longevity bonds (i.e. it can be replicated on the financial market). The independence of $W_\lambda$ and $W_r$ allows us to write the value of the longevity bond as

$$L(t,T) = E^Q_t \left[ p(T) / p(t) \right] E^Q_t \left[ e^{-\int_t^T r(u) du} \right] B(t,T). \quad (6)$$

The behaviour of $L(t,T)$ is described by the differential equation

$$dL(t,T) = r(t)L(t,T) dt + \frac{\partial L(t,T)}{\partial \lambda(t)} \sigma_\lambda dW^Q_\lambda + \frac{\partial L(t,T)}{\partial r(t)} \sigma_r dW^Q_r,$$

and, by letting $\xi_\lambda$ be the market price for the longevity risk, $L(t,T)$ can be rewritten as

$$\frac{dL(t,T)}{L(t,T)} = \left( r(t) + \nabla^L_\lambda \sigma_\lambda \xi_\lambda + \nabla^L_r \sigma_r \xi_r \right) dt + \nabla^L_\lambda \sigma_\lambda dW_\lambda + \nabla^L_r \sigma_r dW_r, \quad (7)$$

where

$$\nabla^L_\lambda \equiv \frac{\partial L(t,T)}{\partial \lambda(t)} \frac{1}{L(t,T)},$$

$$\nabla^L_r \equiv \frac{\partial L(t,T)}{\partial r(t)} \frac{1}{L(t,T)} = \frac{\partial B(t,T)}{\partial r(t)} \frac{1}{B(t,T)},$$

with the last equality immediately following from (6).
All the financial market, accordingly, can be summarized in the following matrix form:

\[
\begin{pmatrix}
\frac{dB}{dt} \\
\frac{dS}{dt} \\
\end{pmatrix} =
\begin{pmatrix}
\begin{bmatrix}
\mu \\
\Sigma \\
\end{bmatrix}
& \begin{bmatrix}
\sigma \xi_r \\
\sigma \xi_\lambda \\
\sigma \xi_S \\
\end{bmatrix}
& \begin{bmatrix}
r(t) + \nabla^2 \sigma_r \xi_r \\
r(t) + \nabla^2 \sigma_\lambda \xi_\lambda + \nabla^2 \sigma_r \xi_r \\
r(t) + \nabla^2 \sigma_S \xi_S + \sigma_S \xi_r \\
\end{bmatrix}
\end{pmatrix}
\begin{bmatrix}
\begin{bmatrix}
dR_r \\
dW_r \\
\end{bmatrix} \\
\begin{bmatrix}
dR_\lambda \\
dW_\lambda \\
\end{bmatrix} \\
\begin{bmatrix}
dR_S \\
dW_S \\
\end{bmatrix}
\end{bmatrix}.
\]

(8)

where \( I_A \) is a diagonal matrix containing the elements of vector \( A \) (i.e. \( B, L, \) and \( S \)).

This financial market is complete since the matrix \( \Sigma \) is invertible. If the longevity bond were not there, \( \Sigma \) would be a matrix \( 2 \times 3 \) and the market wouldn’t be complete: there were not enough assets for hedging against the longevity risk (\( W_\lambda \)).

### 2.3 The investor’s wealth

If we call \( w \in \mathbb{R}^3 \) the vector containing the number of assets \( A \) held in portfolio and \( w_0 \) the number of riskless asset held, then the investor’s wealth (\( R \)) at each instant in time must verify the budget constraint

\[ R = w' A + w_0 G, \]

where the prime denotes transposition. The differential of this constraint is

\[ dR = w' dA + w_0 dG + dw' (A + dA) + dw_0 \cdot G. \]

The two last terms must equate the (opposite of the) investor’s instantaneous consumption because of the self-financing condition while, after substituting for \( w_0 \) taken from the static constraint, \( dR \) can be written as

\[ dR = w' dA + \left( R - w' A \right) \frac{dG}{G} - c dt, \]

and, after easy simplifications,

\[ dR = (Rr + w' I_A (\mu - r I)) - c) dt + w' I_A \Sigma dW, \quad (9) \]

where \( I \in \mathbb{R}^3 \) is a vector containing only 1s.

### 3 The optimal portfolio

In our model we take into account the case of an agent whose preferences are described by a CRRA (Constant Relative Risk Aversion) utility function over his intertemporal consumption. For the sake on simplicity let us now assume
that the agent we are describing is part of the population which the longevity bond is based on. Accordingly, the frequency $p(t)$ can be also viewed as the survival probability of this agent for $t$ periods. Accordingly, the problem can be written as

$$\max_{w,c} \mathbb{E}^p \left[ \int_0^\tau e^{-\rho t} c(t)^{1-\delta} \frac{dt}{1-\delta} \right],$$

where $\tau$ is the death time and $\mathbb{P}$ is the so-called historical probability. The constant parameters $\rho$ and $\delta$ (both strictly positive) measure the subjective discount rate and the relative risk aversion, respectively. In particular, the parameter $\delta$ is assumed not to be lower than 1 (i.e. $1 - \delta < 0$) so that the lower risk aversion is that of the log-investor (in fact when $\delta = 1$ the Problem (10) coincides with the maximization of the log of the investor’s wealth).

Since we assume that $\tau$ is independent of $\mathbb{P}$, then the problem can be rewritten as

$$\max_{w,c} \mathbb{E}^p \left[ \int_0^\tau I_{t<\tau} e^{-\rho t} c(t)^{1-\delta} \frac{dt}{1-\delta} \right],$$

where $I_{t<\tau}$ is the indicator function whose value is 1 when $t < \tau$ and zero otherwise. Since the expected value of an indicator function of any event coincides with the probability of the same event, then the problem is finally written as

$$\max_{w,c} \mathbb{E}^p \left[ \int_0^\infty p(t) e^{-\rho t} c(t)^{1-\delta} \frac{dt}{1-\delta} \right], \quad (10)$$

and the state variables $r(t), \lambda(t),$ and $R(t)$ follow Equations (1), (5), and (9) respectively.

**Proposition 1** Given the state variables described in Equations (1), (5), and (9), the optimal consumption and portfolio solving Problem (10) are

$$c^* = \left( p(t) e^{-\rho t} \right) \frac{R(t)}{F(r,\lambda,t)}$$

\[
\begin{align*}
\frac{Bw^*_B}{R} &= \frac{1}{\delta} \left( \frac{1}{\sigma_\lambda \nabla r} \xi_\lambda + \frac{\sigma_{\lambda S}}{\sigma_\lambda \sigma_S \nabla_\lambda \xi_S} - \frac{\sigma_{S r}}{\sigma_\lambda \sigma_r \nabla_\lambda \xi_r} \right) \\
&\quad + \frac{1}{F(r,\lambda,t)} \left( \frac{1}{\nabla r} \frac{\partial F(r,\lambda,t)}{\partial r(t)} + \frac{1}{\nabla \lambda} \frac{\partial F(r,\lambda,t)}{\partial \lambda(t)} \right),
\end{align*}
\]

\[
\begin{align*}
\frac{Lw^*_L}{R} &= \frac{1}{\delta} \left( \frac{1}{\sigma_\lambda \nabla_\lambda \xi_\lambda} - \frac{\sigma_{\lambda S}}{\sigma_\lambda \sigma_S \nabla_\lambda \xi_S} \right) \\
&\quad + \frac{1}{F(r,\lambda,t)} \frac{1}{\nabla \lambda} \frac{\partial F(r,\lambda,t)}{\partial \lambda(t)},
\end{align*}
\]

\[
\frac{S w^*_S}{R} = \frac{1}{\delta} \frac{\sigma_S}{\sigma_{\lambda S} \nabla_\lambda \xi_S},
\]

\[\text{We recall that } p(0) = 1.\]
where
\[
F(r, \lambda, t) = \mathbb{E}_t^Q \left[ \int_t^\infty e^{-\int_t^s f'_r(\lambda(u) + \rho) du} e^{-\frac{\delta}{2} \int_t^s f'_r(r(u) + \frac{1}{2} \xi') du} ds \right],
\]
\[
dW^Q = \frac{\delta - 1}{\delta} \xi dt + dW.
\]

**Proof.** See Appendix A. ■

As it can be easily checked from Proposition 1, the amount of wealth invested in the longevity bond is entirely deduced from the optimal bond weight. In other words, if the optimal investment in the longevity bond comes out to be 10%, the bond weight must be reduced by 10 basis points.

We state two corollaries to Proposition 1 showing the optimal portfolio for a log-utility agent and for an infinitely risk averse agent. Let us start with the log-utility case.

**Corollary 1** Given the state variables described in Equations (1), (5), and (9), the optimal consumption and portfolio solving Problem (10) for a log-utility consumer (\( \delta = 1 \)) is

\[
e^* = p(t) e^{-\rho t} \frac{R(t)}{F(\lambda, t)}
\]

\[
\frac{Bw^*_B}{R} = \frac{1}{\sigma_r \nabla_r \xi_r} - \frac{\sigma_{Sr}}{\sigma_S \sigma_r \nabla_r \xi_S}
\]

\[
- \frac{1}{\sigma_L \nabla_L \lambda} \xi_L + \frac{\sigma_{LS}}{\sigma_L \sigma_S \nabla_L \lambda} \xi_S - \frac{1}{F(\lambda, t) \nabla_L \lambda} \frac{\partial F(\lambda, t)}{\partial \lambda(t)},
\]

\[
\frac{Lw^*_L}{R} = \frac{1}{\sigma_L \nabla_L \xi_L} - \frac{\sigma_{LS}}{\sigma_L \sigma_S \nabla_L \lambda} \xi_S + \frac{1}{F(\lambda, t) \nabla_L \lambda} \frac{\partial F(\lambda, t)}{\partial \lambda(t)},
\]

\[
\frac{Sw^*_S}{R} = \frac{1}{\sigma_S} \xi_S,
\]

where
\[
F(\lambda, t) = \mathbb{E}_t \left[ \int_t^\infty e^{-\int_t^s f'_r(\lambda(u) + \rho) du} ds \right].
\]

**Proof.** It suffices to put \( \delta = 1 \) in the result of Proposition 1. ■

In this case the function \( F(\lambda, t) \) can be further simplified as

\[
F(\lambda, t) = \mathbb{E}_t \left[ \int_t^\infty p(s) e^{-\rho(s-t)} ds \right] = \mathbb{E}_t \left[ \int_t^\infty \mathbb{E}_t^\tau \left[ I_{s<\tau} e^{-\rho(s-t)} ds \right] \right]
\]

\[
= \mathbb{E}_t^\tau \left[ \int_t^\infty I_{s<\tau} e^{-\rho(s-t)} ds \right] = \mathbb{E}_t^\tau \left[ \int_t^\infty e^{-\rho(s-t)} ds \right],
\]
which is the subjective value of a life annuity paying one monetary unit till the death time $\tau$. Accordingly, we can conclude that the log-utility agent hedges against the relative changes in the subjective value of a life annuity due to the changes in the force of mortality.

The other case we want to show is that of an infinitely risk averse consumer.

**Corollary 2** Given the state variables described in Equations (1), (5), and (9), the optimal consumption and portfolio solving Problem (10) for an infinitely risk averse consumer ($\delta \to \infty$) is

$$c^* = \frac{R(t)}{F(r,t)}$$

$$\frac{Bw_B^*}{R} = \frac{1}{F(r,t)} \frac{\partial F(r,t)}{\partial r(t)}$$

$$\frac{Lw_L^*}{R} = 0,$$

$$\frac{Sw_S^*}{R} = 0,$$

where

$$F(r,t) = E_t^{Q} \left[ \int_t^{\infty} e^{-\int_t^{s} r(u)du} ds \right],$$

$$dW^{Q} = \xi dt + dW.$$  

**Proof.** It suffices to let $\delta$ tending towards infinity in the result of Proposition 1.

The infinitely risk averse consumer buys neither stock nor longevity bonds but just (ordinary) bond proportionally to both its elasticity with respect to the interest rate and the elasticity of a perpetual bond with respect to the interest rate (an analogous result is shown in Wachter, 2003). We just recall that these elasticities coincide with the duration if the instantaneous interest rate $r$ is normally distributed (as in the Merton’s 1970 model).

Furthermore, this kind of investor must consume exactly the number of perpetual annuity he is able to buy time by time with his wealth.

### 4 Conclusion

In this paper we have shown that the presence of a longevity bond allows to complete a financial market. Actually, the longevity bond can be used in order to hedge against the so-called longevity risk (given by the stochastic behaviour of the survival probability).
We take into account the problem of a consumer who maximizes the expected constant relative risk aversion utility of his intertemporal consumption till the date of his death. We demonstrate that the percentage of wealth invested in the longevity bond must be deduced from the wealth invested in the ordinary bond. Accordingly, the presence of a longevity bond on the financial market does not alter the optimal portfolio weights for stocks and riskless assets (or liquidity) since it just affects the investment in bonds.

Furthermore, we show that the optimal hedging component for a log-utility investor is proportional to the relative changes in the subjective value of a life annuity due to the changes in the force of mortality.

Finally, we show that an infinitely risk averse consumer invests neither in stocks nor in longevity bonds but just in ordinary bonds and he consumes, at any instant, the number of perpetual annuity he is able to buy with his wealth.

### A The optimal portfolio

The general problem can be written as

\[
\max_w \mathbb{E}_0 \left[ \int_0^\infty f(z,t) \frac{1}{1 - \delta} c(t)^{1-\delta} \, dt \right]
\]

\[
dz = \mu_z(z,t) \, dt + \Omega(z,t)' \, dW,
\]

\[
dR = (Rr + w'I_AM - c) \, dt + w'I_A \Sigma' \, dW,
\]

where

\[
z \equiv \begin{bmatrix} r \\ \lambda \end{bmatrix}, \quad \mu_z \equiv \begin{bmatrix} \mu_r \\ \mu_\lambda \end{bmatrix}, \quad \Omega' \equiv \begin{bmatrix} \sigma_r & 0 \\ 0 & \sigma_\lambda \end{bmatrix},
\]

\[
M \equiv \begin{bmatrix} \nabla^B r_{\lambda} \xi_r \\ \nabla^L r_{\lambda} \xi_r + \nabla^L r_{\sigma} \xi_r \\ \sigma_S \xi_r + \sigma_{sr} \xi_r \\ \nabla^L r_{\sigma} \xi_r + \nabla^L \sigma_{r} \xi_r \\ \sigma_S \xi_r + \sigma_{sr} \xi_r \end{bmatrix}, \quad \Sigma' \equiv \begin{bmatrix} 0 & 0 \\ \sigma_{sr} & 0 \end{bmatrix},
\]

\[
f(z,t) \equiv p(t) e^{-\rho t}.
\]

The Hamiltonian of this problem is

\[
\mathcal{H} = f(z,t) \frac{1}{1 - \delta} c^{1-\delta} + J_R (Rr + w'I_AM - c) + \mu'_z J_z
\]

\[
+ \frac{1}{2} J_{RR} w'I_A \Sigma' \Sigma A w + \frac{1}{2} tr (\Omega' \Omega J_z z) + w'I_A \Sigma' \Omega J_z R,
\]

from which we can write the first order condition for the optimal consumption and portfolio as

\[
\frac{\partial \mathcal{H}}{\partial w}_{c = c^*} = f(z,t) c^{\ast -\delta} - J_R = 0,
\]

\[
c^* = \left( \frac{J_R}{f(z,t)} \right)^{-\frac{1}{\delta}}.
\]
After substituting the values \( c^* \) and \( w^* \) in the Hamiltonian we obtain the so-called Hamilton-Jacobi-Bellman partial differential equation (HJB):

\[
0 = J_t + \frac{\delta}{1 - \delta} f(z, t)^F + \frac{1}{2} J_{\xi}^2 \Omega \Omega J_{\xi} + \frac{1}{2} tr (\Omega' \Omega J_{\xi}) - \frac{1}{\delta} \frac{1}{2} R(t) R(t)^{1 - \delta} - \frac{1}{2} R(t)^{1 - \delta} \xi' \xi R(t),
\]

with the boundary condition

\[
\lim_{t \to \infty} F(z, t) = 0.
\]
where

\[
\begin{align*}
dZ(s) &= \left( \mu_z - \frac{1}{\delta} \Omega' \xi \right) ds + \Omega' dW, \\
Z(t) &= z.
\end{align*}
\]

In order to switch from the original process

\[dz(s) = \mu_z ds + \Omega' dW,\]

to the new one, it is necessary to define the new probability measure (via the Girsanov theorem)

\[dW^Q = \frac{\delta - 1}{\delta} \xi dt + dW.\]

In our case the vector \(z\) contains two state variables (the survival probability \(p\) and the interest rate \(r\)). Since these two variables are independent, the function \(F(z, t)\) can be written as

\[F(r, \lambda, t) = \mathbb{E}^Q_t \left[ \int_t^\infty e^{-\frac{1}{\delta} \int_s^u (\lambda(u)+p) du} e^{-\frac{1}{\delta} \int_s^u (r(u)+\frac{\delta - 1}{\delta} \xi'} du} ds \right].\]

Now, after substituting the guess function into the optimal consumption and portfolio we obtain

\[
\begin{align*}
c^* &= f(z, t)^\frac{1}{2} \frac{R(t)}{F(z, t)}, \\
I_Aw^* &= \frac{R}{\delta} (\Sigma' \Sigma)^{-1} M + \frac{R}{F} (\Sigma' \Sigma)^{-1} \Sigma' \Omega F_z,
\end{align*}
\]

and, after substituting for the matrices \(\Sigma, M,\) and \(\Omega,\) we finally have the result stated in the proposition.

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