STOCHASTIC OPTIMISATION FOR ALLOCATION PROBLEMS WITH SHORTFALL RISK CONSTRAINTS

by

Monica Billio
Roberto Casarin

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Stochastic Optimisation for Allocation Problems with Shortfall Risk Constraints

Roberto Casarin and Monica Billio

†CEREMADE, Université Paris Dauphine
‡Dept. of Economics, University of Brescia
§GRET A and Dept. of Economics, University of Venice
¶SSAV, School for Advanced Studies in Venice

Abstract
- One of the most important aspects in asset allocation problems is the assumption concerning the probability distribution of asset returns. Financial managers generally suppose normal distribution, even if extreme realizations usually have an higher frequency than in the Gaussian case. The aim of this paper is to propose a general Monte Carlo simulation approach in order to solve an asset allocation problem with shortfall constraint, and to evaluate the exact portfolio risk-level when managers assume a misspecified tails behaviour. In this work we assume, as an example, that returns are generated by a multivariate Student-\( t \) distribution, while in reality returns come from a multivariate skewed Student-\( t \) distribution where each marginal has different degrees of freedom. Stochastic optimisation allows us to value the effective risk for managers. In the specific case analysed, it is also interesting to observe that a multivariate Student-\( t \) with heterogeneous marginal distributions produces a shortfall probability and a shortfall return level that can be approximated adequately by assuming a multivariate Student-\( t \) with a common degree of freedom in the optimisation problem. The present simulation based approach could be an important instrument for investors who need a qualitative assessment of the reliability and sensitivity of their investment strategies when their models are potentially misspecified.

Keywords - Misspecification errors, Portfolio optimisation, Stochastic optimisation, Shortfall constraint, Value at Risk.

J.E.L.: C15, C16, C61, C63, G11.

1 Introduction

In asset allocation problems, assumptions on the probability distribution of future returns are very important aspects. Nevertheless, in many theoretical and empirical works, a normal distribution is usually assumed. It is well known that the normal distribution has
several attractive properties: to produce tractable results in many analytical exercises; all moments of positive order exist, and it is completely characterized by its first two moments, thus establishing the link with the mean-variance optimization theory. Moreover, the Normal distribution arises as the limiting distribution of a whole class of statistical testing and estimation procedures, and therefore plays a central role in empirical modelling exercises.

One of the main characteristics of the normal distribution is the symmetry. Another characteristic is that the distribution tails decay exponentially toward zero; thus extreme realizations are very unlikely. However, as stated by many authors (see Premaratne and Bera [20], Harvey and Siddique [10] and Harvey et al. [9]) these properties seems to contradict empirical findings on asset returns, which evidence that return distribution generally exhibits skewness and leptokurtic behaviour. Leptokurtosis means that extreme returns occur far more often in practice than predicted by the normal model. Left (or positive) skewness implies that the probability of negative returns is higher than that ones of positive returns, as it is often the case for financial returns. Right (or negative) skewness has the opposite meaning and some hedge fund returns can exhibit this negative-skewed distribution.

In a multivariate framework the heterogeneity of the data could also arise in a different way. For financial data to impose the same skewness and fat-tailedness for each component of the return vector seems quite unrealistic. In this paper, we therefore decide to use a non normal distribution and impose different characteristics for each index. In particular we consider the Student-t class and use a rather general multivariate skewed Student-t distribution able to incorporate heterogeneity, fat-tails and skewness features as exhibited by financial data.

For financial managers, who are interested in risk management, these are crucial aspects. A way to deal with the heterogeneity of the data in the decision process is to make use of portfolio models, which account also for the higher moments. In this work, we follow the shortfall probability framework and consider a simple one-period asset allocation problem with a shortfall constraint\(^1\) (see Roy [24], Telser [28], Kataoka [14], Leibowitz and Kogelman [15], Leibowitz, Bader and Kogelman [16], Huisman, Koedijk and Pownall [11] and Kalin and Zagst [13]).

As is known from the safety first principle, the shortfall constraint reflects the investors typical desire to limit downside risk by putting a (probabilistic) upper bound on the maximum loss. In other words, the investor wants to determine an optimal asset allocation for a given Value at Risk (VaR). Results obtained in Lucas and Klassen [17] reveal that the degree of shortfall probability plays a crucial role in determining the effects of the choice between a fat-tailed and a normal distribution. These effects concern the composition of optimal asset allocations as well as the misspecification of the level of kurtosis for the downside risk measure. In our paper we follow and extend the results of Lucas and Klassen [17] by including heterogeneity and skewness and proposing a Monte Carlo approach to perform sensitivity analysis of the optimal allocation to the parameters of the portfolio model. In addition, the optimal allocation problem with shortfall constraint provides a

\(^1\)An extensive literature exists since the fifties, known as Downside Risk Approach. It tries to explain the risk associated to an investment, exclusively evaluating downside oscillations of returns. The Downside Risk is an alternative to the more common standard deviation concept, generally used by financial managers in asset allocation problems. Although these approaches were already developed at the beginning of the fifties, they were followed in the mid seventies with the introduction of the lower partial moment framework (see Harlow [8]). A special case of lower partial moments is the safety first principle (see Roy [24]). This principle allows investors to identify portfolios revealing a minimum probability to fall under a specified return level.
convenient set-up for evaluating the opportunity cost of assuming joint normality returns, while they actually strongly depart from normality.

The first aim of this work is thus to investigate on the role of the shortfall probability in the sensitivity of the optimal allocation to misspecification errors of the return distribution. We show that there exists a critical shortfall probability level. When the desired shortfall probability, say \( \alpha \), is greater than such a critical value, then the assumption of a skewed, fat-tails distribution results in more aggressive asset allocations. As a consequence, if reality is skewed and fat-tailed, optimal asset allocations that are based on the normal distribution may be far too prudent for a given \( (1 - \alpha) \) confidence level VaR. When the shortfall probability is less than this critical level, then to use skewed and leptokurtic distributions leads to more prudent asset allocations. Consequently, an optimal asset mix that is based on a normality assumption, will violate a given \( (1 - \alpha) \) confidence level VaR if reality is skewed and leptokurtic. Moreover, we will look for the relationship between the critical shortfall probability level and the parameters, which are driving the kurtosis and skewness phenomena.

As discussed in Consigli [5], the sensitivity analysis with respect to the parameters of the portfolio model is essential in nonlinear problems with constraints on extreme losses. Many algorithms can be used to solve this kind of problem. In this respect we propose an approach based on simulated stochastic optimisation to study the effects of specification errors. Following the recent literature on stochastic optimisation in finance (see for example Harvey et al. [9]) we apply Monte Carlo simulations and study the misspecification effects in an asset allocation problem with shortfall constraint. We then focus on the effective risk and expected return of an optimal portfolio, when the true distribution of the asset returns is a heterogenous-component multivariate skewed Student-\( t \), while the manager uses a homogenous-components distribution. In fact, it is quite unrealistic to suppose that the probability distribution of each financial class shows the same degrees of leptokurtosis and skewness.

The main result we obtain is that a correct estimate of the degrees of freedom for each component is a necessary condition in order to have no excessive loss of information, an adequate formulation of the optimal strategy and, consequently, a correct perception of the true risk level. We also note that for the particular combination of parameter we consider, it is possible to find a multivariate distribution with identically distributed marginal distributions able to approximate the empirical one, with a loss of information that could be minimal. Being strictly dependent of particular parameter setting under study this result is not general. However, the use of stochastic simulation as suggested is a very effective instrument to analyse the sensitivity of the optimal asset allocation problems to the parameter of the assumed return distribution.

The structure of the work is as follows. In Section 2 we present the asset allocation problem and the class of skew Student-\( t \) distributions. In Section 3 we examine how the problem can be solved through a Monte Carlo approach and concentrate on the stochastic optimization aspects. After this analysis, we give some general characterizations of the theoretical effects of fat tails and skewness on the problem at hand (see Section 4). The numerical analysis of the effects of misspecification on the downside risk is in Section 5, when using identically distributed returns of three financial assets class (cash, stocks and bonds) and in Section 7, when using heterogenous marginal distributions.
The portfolio model

We consider a one-period model with \( n \) asset categories. At the beginning of the period, the manager can invest the available funds in any of the \( n \) asset categories and short positions are not allowed.

The objective of the investment manager is to maximize the expected return on the portfolio, subject to a shortfall constraint. This shortfall constraint states that with a sufficiently high probability \( (1 - \alpha) \), the return on the portfolio will not fall below the threshold return \( r_{\text{low}} \).

Formally, the asset allocation problem can be written as follows

\[
\begin{align*}
\text{Max} & \quad \mathbb{E} \left( \sum_{i=1}^{n} x_i r_i \right) \\
\mathbb{P} \left( \sum_{i=1}^{n} x_i r_i < r_{\text{low}} \right) & \leq \alpha \\
\sum_{i=1}^{n} x_i & = 1 \\
x_i & \geq 0
\end{align*}
\]

where \( x = (x_1, \ldots, x_n)' \) and \( x_i \) and \( r_i \), (with \( i = 1, \ldots, n \)), denote the fraction of capital invested in the asset category \( i \), and the (stochastic) return on asset category \( i \), respectively. The operator \( \mathbb{E}(\cdot) \) is the expectation operator with respect to the probability distribution \( \mathbb{P} \) of the asset returns. The probabilistic constraint in (2) fixes the permitted VaR for feasible asset allocation strategies. We know that VaR is the maximum amount that can be lost with a certain confidence level in a given period. In the setting of (2) with \( r_{\text{low}} < 0 \), the VaR per Euro invested is \( -r_{\text{low}} \) with a confidence level of \( (1 - \alpha) \).

Our aim is to use a stochastic optimisation technique to study the asset allocation problem in Equations (1) to (4), when asset returns follow general multivariate distributions. More precisely, we need a class of probability distributions that allows for fat tails and skewness. The class of multivariate skewed Student-\( t \) distributions meets these requirements.

A huge amount of literature focused on the univariate skewed Student-\( t \). One of the first definition of multivariate skewed Student-\( t \) distribution is due to Azzalini and Dalla Valle [1]. Their results are then generalised by Branco and Dey [3], which introduce the class of skewed elliptical distributions. Another generalisation of the class of elliptical skewed is due to Sahu et al. [27].

In the following we adopt a definition of multivariate skewed based on the linear transform of univariate skewed Student-\( t \) distributions. This technique of construction of multidimensional distribution was proposed first in Bawuens and Laurent [2]. More precisely, we follow the definition due to Ferreira and Steel [7]. Their constructive method for multivariate skewed Student-\( t \) has many advantages. It is simple to simulate from, the existence of the moments is guaranteed by the existence of the moments of the underlying univariate distributions and the resulting multivariate distribution allows for different magnitudes and directions of kurtosis and skewness.

Let the function

\[
f(r; \nu, \gamma) = \prod_{j=1}^{n} g(r_j; \nu_j, \gamma_j), \quad \text{with } \gamma_j > 0,
\]

where
the density of a multivariate skewed distribution, $SkT_n(r;0,I_n,\nu,\gamma)$, with independent components, where

$$
g(r;\nu,\gamma) = \frac{2}{\gamma + 1} \left\{ h_\nu \left( \frac{r}{\gamma} \right) \mathbb{I}_{[0,\infty)}(r) + h_\nu \left( \frac{r\gamma}{\nu} \right) \mathbb{I}_{(-\infty,0)}(r) \right\},
$$

$$
h_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)(\pi\nu)^{1/2}} \left( 1 + \frac{x^2}{\nu} \right)^{-(\nu+1)/2} \mathbb{I}_{(-\infty,\infty)}(x), \text{ with } \nu > 2.
$$

In the previous equations, $h_\nu$ is the density of a univariate symmetric Student-$t$, $\mathbb{I}_A(x)$ is the indicator function, $\Gamma(\cdot)$ denotes the gamma function, $r = (r_1,r_2,\ldots,r_n)' \in \mathbb{R}^n$ represents the random vector of asset returns, $\nu = (\nu_1,\ldots,\nu_n)' \in (2,\infty)^n$ is the vector of degrees of freedom and $\gamma = (\gamma_1,\ldots,\gamma_n)' \in \mathbb{R}^n_+$ is the vector of skewness parameters.

Let $\mu \in \mathbb{R}^n$ and $\Omega \in \mathbb{R}^{n \times n}$ be a mean vector and a non-singular matrix (called precision matrix) respectively. The following linear affine transformation

$$
u = \Omega^{1/2} r + \mu,
$$

allows us to introduce dependence among the components of the random vector $r \sim SkT_n(r;0,I_n,\nu,\gamma)$. The resulting vector $\nu$ follows a multivariate skewed Student-$t$ distribution, that is denoted with $SkT_n(r;\mu,\Omega,\nu,\gamma)$.

The first two moments of the skewed Student-$t$ distribution play an important role in the subsequent analysis. The first and second order moments of the distribution can be explicitly calculated. Let us consider a vector $r$ of independent standard skewed Student-$t$. The $i$-th components of the mean and of the variance vector are

$$
\mathbb{E}(r_i) = \frac{\gamma_i^2 - 1}{\gamma_i} \frac{\nu_i - 1}{\nu_i/2} \left( \frac{\nu_i}{\pi} \right)^{1/2}
$$

$$
\text{Var}(r_i) = \frac{\nu_i^4 - \gamma_i^2 + 1}{\gamma_i^2} \left( \frac{\nu_i}{\nu_i - 2} - \frac{\nu_i}{\pi} \left( \frac{\Gamma((\nu_i - 1)/2)}{\Gamma(\nu_i/2)} \right)^2 \right) + \frac{\nu_i}{\pi} \left( \frac{\Gamma((\nu_i - 1)/2)}{\Gamma(\nu_i/2)} \right)^2,
$$

respectively. For further details see Appendix A.

The result extends easily to the dependent case. In particular the variance-covariance matrix of the random vector $\nu \sim SkT_n(\mu,\Omega,\nu,\gamma)$ is

$$
\text{Var}(\nu) = \Omega^{1/2} \text{Var}(r) \Omega^{1/2} = \Omega^{1/2} \Xi(\nu,\gamma) \Omega^{1/2}
$$

where $\Xi = \text{diag}\{\xi_1,\ldots,\xi_n\}$ and $\xi_i = \text{Var}(r_i)$.

Fig. 1 shows the effects of the parameters $\nu$ and $\gamma$ on the precision and on the tails of the univariate skewed Student-$t$. The distributions are standardized in such a way that they all have zero mean and unit variance. We observe that the lower is the values of $\nu$ the fatter are the tails and more peaked around the mean is the distribution. As the values of $\gamma$ decreases, with $\gamma < 1$, the left tail becomes fatter and the distribution become more peaked around the mode.

In the following examples we focus on some bivariate special cases of multivariate skewed Student-$t$. The main aim is to show the effect of the heterogeneity on the distribution.

**Example 2.1 - Homogeneous-components multivariate symmetric Student-$t$**. Let us set $\gamma_i = 1$, and $\nu_i = \nu$, $\forall i = 1,\ldots,n$, in Eq. 5, then we retrieve the symmetric multivariate Student-$t$ distribution with homogeneous components. An example of bivariate symmetric
Figure 1: Upper chart shows Skewed Student-t for different values of $\gamma$. Bottom chart shows symmetric Student-t for different values of $\nu$.

Figure 2: Contours of standard bivariate Student-t distributions for different magnitudes of kurtosis and skewness. Symmetric Student-t with homogeneous components (top-left chart) and symmetric Student-t with heterogeneous degrees of kurtosis: $\nu_1 = 3$, $\nu_2 = 40$ (top-right chart); asymmetric Student-t, with heterogeneous skewness parameters: $\gamma_1 = 1$, $\gamma_2 = 0.8$ and constant kurtosis: $\nu_1 = \nu_2 = 3$ (bottom-left chart); asymmetric Student-t, with heterogeneous skewness parameters: $\gamma_1 = 1$, $\gamma_2 = 0.8$ and heterogeneous degrees of freedom: $\nu = 3$, $\nu = 40$, (bottom-right chart).
Student-$t$ with homogeneous components is given in Fig. 2. The distribution has zero mean and unit precision matrix. The degrees of freedom parameter $\nu$ determines the degrees of leptokurtosis of the Student-$t$ distribution in all the directions of the support set. The smaller $\nu$, the fatter the tails of the distribution. The Student-$t$ distribution has the normal distribution as special case if $\nu \to \infty$. The variance-covariance matrix given in Eq. 11 reduces to

$$\text{V}(r) = \nu \Omega / (\nu - 2)$$

and requires $\nu > 2$. Note the the variance depends on $\nu$.

\[ \square \]

**Example 2.2 - Homogeneous-components multivariate skewed Student-$t$.**

A multivariate Student-$t$ is homogeneous when the degrees of kurtosis $\nu$ and the skewness $\gamma$ are constant over the components. Consider now $u \sim SkT_n(\mu, \Omega, \nu_i, \gamma_i)$, with $i = (1, \ldots, 1)' \in \mathbb{R}^n$, the relation between the variance-covariance matrix and the precision matrix $\Omega^{-1}$ is

$$\text{V}(u) = \left( \frac{\gamma^4 - \gamma^2 + 1}{\gamma^2} \left( \frac{\nu}{\nu - 2} - \frac{\nu}{\pi} \left( \frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} \right)^2 \right) + \frac{\nu}{\pi} \left( \frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)} \right)^2 \right) \Omega.$$  

(13)

The existence of the second order moment is again guaranteed by the condition $\nu > 2$. It is important to note that the variance depends now on both of the parameters $\nu$ and $\gamma$. When $\gamma = 1$ Eq. 13 reduces to Eq. 12. An example of bivariate skewed distribution with homogeneous independent component is given in the bottom-left chart of Fig. 1.

\[ \square \]

**Example 2.3 - Heterogeneous-components multivariate Student-$t$.**

In this work we deal with heterogeneous-components distributions. In a multivariate context the asymmetry and the degrees of kurtosis can be differentiated in each direction. The charts at the bottom of Fig. 2 exhibit two examples of bivariate densities with heterogeneous components. The bottom-left chart shows the level set of a density with direction-specific skewness ($\gamma_1 = 1$, $\gamma_2 = 0.8$) and constant kurtosis ($\nu_1 = \nu_2 = 3$). The bottom-right chart shows the level set of a density with direction-specific skewness ($\gamma_1 = 1$, $\gamma_2 = 0.8$) and kurtosis ($\nu_1 = 3$, $\nu_2 = 40$). In this work we also consider symmetric distributions with component-specific kurtosis level. A bivariate example of such a densities is given in the upper-right chart of Fig. 2.

\[ \square \]

In the next sections we present a Monte Carlo approach to analyse an asset allocation problem with shortfall constraint. The proposed simulation-based approach is general enough to deal with the general class of heterogeneous-component skewed Student-$t$ distributions.

### 3 Simulation based stochastic optimisation

We use a Monte Carlo simulation approach (see Ripley [21], Rubinstein [25] and Niederreiter [18]) in order to solve the optimization problem and to compute numerically nontrivial integrals. There are many features that distinguish this method from most of the others generally used. First, in spite of its robustness and simplicity it can handle problems of far greater complexity and size than most other methods. Second, the probabilistic nature of
stochastic simulation allows us to apply central limit theorem and law of large numbers. Thus, the accuracy of the Monte Carlo method can be controlled by adjusting the sample size.

To simulate assets returns from a skewed Student-
\textit{t} distribution with \( \nu \) degrees of freedom and skewness \( \gamma \) we use the following stochastic representation. Let \( Z \) have a Bernoulli distribution with parameter \( 1/(\nu^2 + 1) \) and \( X \) a Student-
\textit{t} distribution with \( \nu \) degrees of freedom\(^2\) then

\[ Y = Z|X|\gamma - (1 - Z)|X|\frac{1}{\gamma}, \]

follows a skewed distribution with skewness parameter \( \gamma \).

The Monte Carlo method allows numerical solutions of the following \textit{stochastic optimisation problem}

\[ x^* = \arg \max_{x \in U} \mathbb{E}\{g(x, Z)\}, \quad (15) \]

where \( Z \) is a random variable with p.d.f. \( h(z) \). The solution of the problem (15) needs the computation of \( \mathbb{E}(\cdot) \).

The most simple numerical solution approach (see Judd [12] for some examples of application in economics, Robert [22], Robert and Casella [23] for an introduction to other stochastic optimisation methods) is to take a sample of size \( D \) of i.i.d. random variables \( Z_i \sim h(z) \), and to approximate \( \mathbb{E}(g(x, Z)) \) by its sample mean

\[ \frac{1}{D} \sum_{i=1}^{D} g(x, Z_i). \quad (16) \]

Then all standard optimisation techniques can be applied.

We apply this approach to solve our portfolio problem. In fact the use of Monte Carlo integration is quite natural for such problems since we are essentially simulating the problem. The solution is denoted by \( \tilde{x}^* \) and approximates the true solution \( x^* \). It gives us the optimal fraction of capital which has to be invested in each asset class to maximise expected return.

The quality of this procedure depends on the size \( D \) and how well the integral is approximated by the random sample mean. We are therefore interested in knowing the sample size \( D \) and the number of samples \( N \) of \( D \) draws, necessary to obtain a "good" estimate. To do that, we control the error from the analytical solution\(^3\) and the standard deviation of each estimate. In our context we have seen that for \( D = 10,000 \) and \( N = 100 \), we can approximate the underlying distribution adequately and obtain an accurate estimate with a very small standard error\(^4\).

\(^2\)Let \( V \) follow a standard Gaussian distribution and \( W \) a chi-square distributions with \( \nu \) degrees of freedom and let \( V \) and \( W \) be independent, then

\[ X = \frac{V}{\sqrt{W/\nu}}, \quad (14) \]

has a Student-
\textit{t} distribution with \( \nu \) degrees of freedom.

\(^3\)For multivariate normal or Student-
\textit{t} distribution, the \textit{analytical solution} is obtained by calculating integrals "analytically" instead of "numerically".

\(^4\)Although there exist analytical techniques to choose the optimal number of simulation \( N \) (see Rubinstein [25]), graphical techniques are often preferred (see Robert [22], Robert and Casella [23]). They permit the choice of the adequate simulations number, necessary to obtain the stabilisation of the solution. The higher \( N \), the better the solution approximation, because the variance of sample mean reduces to zero.
4 Theoretical Aspects

The parameters ν and γ play a prominent role in our asset allocation problem, given their presence in the shortfall constraint. Moreover the precision matrix (see Eq. 13) depends on both ν and γ if we hold fixed the variance of the returns. Decreasing ν has two effects. First of all, the tails of the distribution become fatter, resulting in a higher probability of extreme events for fixed precision matrix Ω⁻¹ and skewness level γ. Secondly as ν decreases, the eigenvalues of the precision matrix increase, for all values of γ, thus the distribution becomes more concentrated around the mean.

The skewness parameter γ affects the whole probability distribution. If we hold fixed the variance, note that as γ decreases, the eigenvalues of the precision matrix Ω⁻¹ increase for γ < 1 and decrease for γ > 1. As with regard to the effect of γ on the tails, when the distribution is skewed to the left (γ < 1), the fatter-tails effect on the shortfall constraint becomes stronger. When the distribution is skewed to the right (γ > 1), the left tail is thinner than as it was in the symmetric case. This reduces the intensity of the fatter-tails effect on the shortfall probability.

The composite effect on the shortfall constraint of altering ν and γ depends critically on the shortfall probability α. It can be shown that for a sufficiently small value of α, the shortfall constraint becomes less binding if the distribution used tends to the normal (ν → +∞ and γ → 1). The reverse holds if we consider sufficiently large values of α.

It is interesting to present the break-even shortfall probability for the normal distribution, i.e., the value of α, as a function of ν and γ, such that the shortfall constraint is as binding as the shortfall constraint for the corresponding normal distribution (see Lucas and Klaassen [17]). More precisely, let \( u \sim N(0, 1) \) and \( w \sim SkT(0, 1, \nu, \gamma) \) be univariate random variables with p.d.f. \( f \) and \( g \) respectively. For the two distributions, the quantile associated to the shortfall probability α are

\[
\Phi_N(r^\text{low}_N) \triangleq \int_{-\infty}^{r^\text{low}_N} f(u; 0, 1)du = \alpha \quad \Leftrightarrow \quad r^\text{low}_N = \Phi_N^{-1}(\alpha) \quad (17)
\]

\[
\Phi_{SkT}(r^\text{low}_{SkT}) \triangleq \int_{-\infty}^{r^\text{low}_{SkT}} g(w; 0, 1, \nu, \gamma)dw = \alpha \quad \Leftrightarrow \quad r^\text{low}_{SkT} = \Phi_{SkT}^{-1}(\alpha; \nu, \gamma), \quad (18)
\]

where \( \Phi_{SkT} \) and \( \Phi_N \) denote the Student-t and normal c.d.f. respectively.

In order to obtain the critical shortfall probability \( \alpha(\nu, \gamma) \), for fixed values of ν and γ we set

\[
r^\text{low}_N \xi^{1/2}(\nu, \gamma) + \theta(\nu, \gamma) = r^\text{low}_{SkT},
\]

where \( \xi(\nu, \gamma) \) and \( \theta(\nu, \gamma) \) are defined in Appendix A. We then solve w.r.t. α the resulting nonlinear equation

\[
\theta(\nu, \gamma) + \xi^{1/2}(\nu, \gamma)\Phi_N^{-1}(\alpha) = \Phi_{SkT}^{-1}(\alpha; \nu, \gamma). \quad (19)
\]

Such a value of α produces the same optimal allocation under either normality assumption or the assumption of skewed Student-t distributed returns with ν degrees of freedom and skewness parameter γ. The values of \( \alpha(\nu, \gamma) \) as function of ν are given in Figure 3, for different levels of γ. This graph presents the critical shortfall probability which has range from \( \alpha = 1.79\% \) for \( \nu = 3 \) to \( \alpha = 3.89\% \) for \( \nu = 20 \), when γ = 1. For γ < 1 the critical value \( \alpha(\nu, \gamma) \) increases for all ν, and for γ > 1, the critical probability level decreases for all ν.

For values of α below these critical levels, the effect of fat tails on the shortfall constraint dominates the effect caused by increased precision. In these cases, the probability
restriction in (2) for the skewed Student-$t$ distribution is more binding for a given asset allocation than in the case of normally distributed asset returns. Again, the reverse holds for values of $\alpha$ above the critical level. In our empirical study, we use $\alpha = 0.5\%$, $\alpha = 1\%$, $\alpha = 5\%$ and $\alpha = 10\%$ in order to illustrate both settings.

5 Results for Homogenous-components Student-$t$

To illustrate our findings, we solve through a Monte Carlo approach a strategic asset allocation problem (similar to which studied also by Lucas and Klaassen [17]). Strategic allocation decisions refer to long-term investments and usually involve a small number of assets, which represent the benchmarks for different asset categories. Due to the increasing volatility of the financial markets, long-term investors require constraints on the portfolio’s downside risk. In this context, our simulation-based sensitivity analysis is not too computational demanding and is a valuable tool for managers, which are facing that kind of constraints and want study the impact of misspecification errors on the effective risk of the managed portfolio.

In the following we consider three U.S. asset categories: cash, stocks, and bonds. For cash, we use the return on one month Eurodollar deposits. Stock returns are based on the S&P 500 and include dividends. Bond returns are computed using holding period returns on 10-year Treasury bonds. In order to compare the results with that ones given in Lucas and Klaassen [17], we consider annual returns over the period 1983-1994. All data are obtained from Datastream. Initially, we need to compute the mean and variance of the returns series. Let $x_1$, $x_2$ and $x_3$ denote the amount invested in cash, stocks, and bonds, respectively, and let the corresponding returns be denoted by $r_1$, $r_2$ and $r_3$. Then
\( r = (r_1, r_2, r_3)' \) has mean and standard deviation

\[
\begin{align*}
\mu &= (6.8\%, \ 17\%, \ 12.3\%) \\
\sigma &= (2.3\%, \ 14.7\%, \ 10.5\%)
\end{align*}
\]

and the correlation matrix

\[
\begin{pmatrix}
1 & 0.01 & 0.18 \\
0.01 & 1 & 0.73 \\
0.18 & 0.73 & 1
\end{pmatrix}
\]

In Table 1 we consider results obtained for two values of shortfall probability \((\alpha = 0\%, \ \alpha = 5\%)\) and for several values of the shortfall return \(r^{\text{low}}\) at the levels of 0\%, −5\%, −10%.

Thus, for example, the combination \((\alpha, r^{\text{low}}) = (1\%, 0\%)\), means that the manager requires an asset mix that results in no loss with a 99\% probability. Similarly, the combination \((\alpha, r^{\text{low}}) = (5\%, -5\%)\), means that the manager is satisfied with a 5\% VaR per Euro invested with a confidence level of 95\% probability.

Using GAUSS optimisation library, we compute in simulation the optimal values of \(x_i\) satisfying the shortfall constraint in (2) for several values of \(\nu\). The main results are presented in Table 1.

Some obvious effects in Table 1 are that the optimal asset mixes become more aggressive if the shortfall constraint is loosened. This can be done by increasing the allowed shortfall probability \(\alpha\) or by lowering the required shortfall return \(r^{\text{low}}\), i.e. increasing the VaR per Euro invested. If we focus on the effects of \(\nu\), we note the difference between the \(\alpha = 5\%\) and the \(\alpha = 1\%\) case. In the 5\% case, increasing the fatness of the tails of the asset return distribution \(P\) then leads to more aggressive asset allocations. In the 1\% case, an increase in \(\nu\) leads to less aggressive allocations.

This result is easily understood, given what we have seen analysing Figure 3. In fact, decreasing \(\nu\) while keeping \(\gamma\) and the variance fixed has two opposite effects. First, the probability of extreme (negative) returns increases, leading to more prudent asset allocation strategies. Second, the precision of the distribution increases, leading to more certainty about the spread of the outcome and, thus, to a more aggressive strategy. For a shortfall probability of 5\%, the latter of these two effects dominates, while for a probability of 1\% the opposite.

The above results clearly depend on the value of \(\gamma\). As showed in Fig. 3, when \(\gamma < 1\) the critical shortfall probability is greater than it was in the symmetric case. As usual values of \(\alpha\) greater than the critical level leads to aggressive allocations. The reverse holds for values of \(\alpha\) less than the critical level.

Figure 4 shows the effects on expected portfolio returns, for different distributions, when the shortfall probability varies.\(^6\) Curves obtained by simulations, are labelled: \textit{Shortfall Probability Efficient Frontiers (SPEF)}\(^7\) (see Rudolf [26]), and represent all the efficient portfolios, given a certain level of risk (expressed by the shortfall probability).

\(^5\)The studies we made use in particular four values of shortfall probability \((0.5\%, 1\%, 5\%, 10\%)\), and five values of shortfall return \((0\%, -3\%, -5\%, -7\%, -10\%)\). Table 1 only indicates the main results obtained in our work.

\(^6\)\(r^{\text{low}}\) has been set equal to 0\%.

\(^7\)They are an alternative representations of the Efficient Frontier, known from the Portfolio theory.
Table 1: Optimal asset allocation obtained through Monte Carlo simulation ($E^*$ indicates the expected portfolio returns).

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$r_{low} = 0%$</th>
<th>$r_{low} = -5%$</th>
<th>$r_{low} = -10%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cash Stock Bond $E^*$</td>
<td>Cash Stock Bond $E^*$</td>
<td>Cash Stock Bond $E^*$</td>
</tr>
<tr>
<td>Shortfall probability 5%</td>
<td>Shortfall probability 1%</td>
<td>Shortfall probability 1%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>28.10% 66.70% 5.10% 13.90%</td>
<td>80.80% 19.20% 0.00% 8.70%</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>46.30% 50.10% 3.50% 12.10%</td>
<td>80.00% 20.00% 0.00% 8.80%</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>49.00% 48.50% 2.50% 11.90%</td>
<td>78.50% 21.50% 0.00% 9.00%</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>50.30% 47.20% 2.50% 11.70%</td>
<td>77.70% 22.30% 0.00% 9.10%</td>
<td></td>
</tr>
<tr>
<td>$\infty$</td>
<td>51.60% 45.90% 2.50% 11.60%</td>
<td>75.60% 24.40% 0.00% 9.30%</td>
<td></td>
</tr>
</tbody>
</table>

freedom. The reverse holds on the right side. In particular, the SPEF obtained from the normal distribution intercepts all the others in different points for a level of $\alpha$ that is the same depicted in Figure 3 for $\gamma = 1$.

Through the Monte Carlo study, it is also possible to determine the effects of a variation of the shortfall probability on the fractions invested in each asset. If we observe Figure 7 (Appendix B), we can note that the higher the level of $\alpha$, the lower the fraction invested in cash, since to obtain more aggressive portfolios the manager directs capitals in riskier assets. However, while for a sufficiently small value of $\alpha$ (i.e. 0.5%, 1%), a normal distribution shows lower value invested in cash than the other distributions, for higher levels of shortfall probability (i.e. 5%, 10%), the lowest percentages in liquid assets are obtained augmenting the degrees of leptokurtosis. The reverse holds for stocks. In fact for this asset category, the behaviour is exactly reverse to which showed by cash, since there exists a trade-off between liquid and risky asset in order to obtain efficient portfolios. Fractions invested in bonds shows instead a more complex behaviour. In general, we can say that initially the percentage is increasing for smaller value of shortfall probability, and it is decreasing for higher value of shortfall probability.

The intersection between the curve of the normal distribution and the others in this case too, occurs for the values of $\alpha$ showed in Figure 3. However, in this particular case each curve intercepts the others in more than one point, so they are more difficult to analyse.

It is also interesting to observe the expected returns behaviour when the shortfall return varies (given the shortfall probability level). We have choose to graph the expected
returns behaviour for $\alpha = 1\%$ and for $\alpha = 5\%$. In Appendix C we can note that the higher the shortfall return value, the lower the expected return, but while for $\alpha = 1\%$ the fat tail effect dominates, for an $\alpha = 5\%$ the highest values are obtained when asset returns are leptokurtic. For very high losses levels (ex. 10%), it makes no difference if we use the normal distribution or a Student-$t$ with different degrees of freedom, since the portfolio always contains only risky assets (100% stocks).

We can carry out a similar analysis, observing the effects on the fractions invested in each asset, varying $r^{low}$ for different levels of shortfall probability. The results are showed in Appendix D.

6 Effects of misspecified tail behaviour

The probability distribution $\mathbb{P}$ is taken by the investment manager as a description of the true distribution of the asset returns. We label the distribution used by the manager $\mathbb{P}_m$ and the true distribution $\mathbb{P}_t$. These distributions are characterized by the parameter specifications $(\mu_m, \Omega_m, \nu_m)$ and $(\mu_t, \Omega_t, \nu_t)$ respectively.

Obviously, the manager would do best by matching $\mu_m, \Omega_m$ and $\nu_m$ to $\mu_t, \Omega_t$ and $\nu_t$, respectively. However, the manager can fail to match all the parameters of the distribution used to solve (1) and (2) to those of the true distribution $\mathbb{P}_t$.

The effects of misspecification of means $\mu_m$ and/or covariance $\Omega_m$ has been investigated in the literature (see, e.g. Chopra and Ziemba [4]). We concentrate here on the possible mismatch between the true degree of leptokurtosis and the degree of leptokurtosis used by the investment manager, while assuming that means and covariance of the returns are precisely estimated.

The most obvious example of such a situation is the use of the normal distribution for solving (1), while the asset returns are actually fat-tailed. The mismatch between $\nu_m$ and $\nu_t$ can have important effects for the feasibility and efficiency of the optimal asset mixes. We assume that for a given value of $\nu_m$, the manager chooses $\mu_m$ and $\Omega_m$, in such
Figure 5: True Shortfall Probability for several combinations of $\nu_m$ and $\nu_t$, obtained by simulation.

a way that the mean and variance of $P_m$ match the corresponding moments of the true distribution $P_t$. This amounts to setting $\mu_m = \mu_t$ and

$$\Omega_m = (1 - 2\nu_m^{-1}) \frac{\nu_t \Omega_t}{\nu_t - 2}. \tag{22}$$

We assume, that the true mean $\mu_t$ and variance $\nu_t \Omega_t / (\nu_t - 2)$ are observed without error. We abstract from estimation error for exposition purposes and in order to fully concentrate on the effects of fat tails (the results would be very similar if slightly different values for the means and variances were used, thus allowing for estimation error). Let $x_m$ denote the optimal strategy of the investment manager using the distribution $P_m$ with $\nu_m$ degrees of freedom. The appropriate values of $x_m$ can be found in Table 1.

We want to compute the effect of using $x_m$ when the data follow the distribution $P_t$ instead of $P_m$. In particular, we are interested in the effect of a discrepancy between $\nu_m$ and $\nu_t$ on the shortfall constraint. We can quantify this effect in at least two different ways.

First keep $r^{low}$ fixed and compute

$$\alpha^* = P_t \left( \sum_{i=1}^{3} x_{m,i} (1 + r_i) \leq 1 + r^{low} \right), \tag{23}$$

where $x_{m,i}$ is the optimal asset allocation to category $i$ for $\nu_m$ (see Table 1). We simulate the asset returns from a distribution with $\nu_t$ degrees of freedom and evaluate the shortfall constraint under the strategy $x_m$. We find the true confidence level $(1 - \alpha^*)$ of the investment manager’s VaR.

The results for $r^{low} = 0\%$ are given in Figure 5. Different values for $r^{low}$ produce similar results. The right panel in the figure gives the results if the optimal strategy is computed with $\alpha = 5\%$. First note that, the true shortfall probability $\alpha^*$ is equal to $\alpha$, if and only if the investment manager uses the correct distribution, i.e., $\nu_m = \nu_t$.

Second, if the investment manager uses a distribution that has thinner tails than those of the true distribution, then the manager is conservative in the sense that the shortfall constraint in (2) is not binding.

This holds even though the manager may believe the constraint to be binding based on the (misspecified) distribution $P_m$ of the asset returns. As a result, efficiency could be
Table 2: Differences \((r^{*,\text{low}} - r^{\text{low}})\) in basis points.

<table>
<thead>
<tr>
<th>(\nu_m)</th>
<th>(r^{\text{low}} = 0%)</th>
<th>(r^{\text{low}} = -5%)</th>
<th>(r^{\text{low}} = -10%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\nu_t)</td>
<td>(\nu_t)</td>
<td>(\nu_t)</td>
<td>(\nu_t)</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>(\infty)</td>
<td>3</td>
</tr>
<tr>
<td><strong>Shortfall probability 5%</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-249</td>
<td>-296</td>
</tr>
<tr>
<td>7</td>
<td>178</td>
<td>0</td>
<td>-33</td>
</tr>
<tr>
<td>(\infty)</td>
<td>201</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td><strong>Shortfall probability 1%</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>43</td>
<td>88</td>
</tr>
<tr>
<td>7</td>
<td>-45</td>
<td>0</td>
<td>51</td>
</tr>
<tr>
<td>(\infty)</td>
<td>-105</td>
<td>-60</td>
<td>0</td>
</tr>
</tbody>
</table>

The differences \((r^{*,\text{low}} - r^{\text{low}})\), in basis points, obtained in simulation, are presented in Table 2. The value of \(r^{*,\text{low}}\) in (24), is the managers true VaR if the investment policy \(x_m\) based on \(P_m\) is used.

The qualitative results are similar to those in Figure 5. For high values of \(\alpha\), using a distribution \(P_m\), which has thin tails compared to reality \(P_t\), produces a conservative strategy. Again, the opposite holds for small values of the shortfall probability \(\alpha\).

The impact of using the normal distribution for \(P_m\) if reality is fat-tailed is quite substantial. Assume the postulated required minimum return \(r^{\text{low}}\) is \(-5\%\), i.e., a VaR of 5 cents per Euro invested. That is, with a maximum probability of \(\alpha\), the manager is willing to take losses exceeding 5% of the invested notional principal.

If \(\alpha = 5\%\), the true shortfall return can be as much as 352 basis points above the postulated level, implying a shortfall return of about \(-1.45\%\) instead of \(-5\%\), with a probability of 95%. Exploiting the fat-tails property in this case can lead to more aggressive asset allocations and, therefore, efficiency gains for a given level of shortfall. Alternatively the case \(\alpha = 1\%\) lead to a violation of the shortfall constraint.

7 Results for Heterogeneous-components Student-\(t\)

As we know, the hypothesis that the returns of different asset classes are identically distributed is quite unrealistic, given the heterogeneity of the financial indexes studied in this
work.

We therefore decide to develop our analysis, using a different probability distribution for each index, remaining in the Student-t class. The use of stochastic optimisation techniques allow us to extend the study of the asset allocation problems with multivariate distribution for which in general it is not possible to obtain analytical results. This approach makes it necessary to determine which Student-t combination seems able to fit the empirical distribution adequately. To do this, we have analysed, through several tests\(^8\), the behaviour of some distributions of financial indexes\(^9\), for each asset category (cash, stocks, bonds), valuing for each of them the existence of *fat-tails*, and the parametric distribution that allows us to obtain the best fit for the empirical distribution on the tails. The test results show that the normal distribution is often unable to capture the behaviour of the tails, since financial time series are usually leptokurtic, while the Student-t distribution seems more adequate to capture the fat-tails effect.

In particular, tests indicate that for stocks, the Student-t with 7 degrees of freedom seems a good approximation of the empirical distribution in the extreme returns area. Otherwise, bonds seem to prefer Student-t with 8 degrees of freedom. Finally, cash behaviour does not appear unimodal, and for this reason no Student-t appears suitable to fit the empirical distribution adequately. Yet, some statistical tests seem to accept the null hypothesis that some of the cash indexes studied follow a Student-t distribution with 30 degrees of freedom (well approximated by a normal). For this reason we use this marginal distribution to simulate returns even if we know that it is not the most suitable. In fact our objective is to study the effects on the portfolio model when the empirical distribution is fat-tails, and in particular when each asset class presents a different degree of leptokurtosis. To do this, we generate asset returns simulating from the Student-t class, which present leptokurtic but not plurimodal or asymmetric characteristics.

Making use of tests results, we now calculate the optimal investment strategy and expected portfolio returns, generating asset returns from a multivariate distribution\(^10\), where marginal distributions are \(t(30)\), \(t(7)\) and \(t(8)\). We label this heterogeneous-components distribution as "Mixed" with mean \(\mu\) and variance-covariance matrix \(V\). To impose the desired correlation structure to the simulated series, we use the *calibration method*.

In particular, given two variables \(x\) and \(w\), where \(x \sim t(\nu_x)\) and \(w \sim t(\nu_w)\), we say that the correlation between \(x\) and \(w\) is \(\rho^*\) if there exists a value \((\tau^*)\) for the parameter \(\tau\) such that \(\rho^* = f(\tau^*, \nu_x, \nu_w)\), where \(f\) is the correlation between the components of the random vector given in (14). It is very difficult to obtain an analytical solution, and therefore we use the Monte Carlo method to calibrate \(\tau\) step by step, until we obtain the desired value for \(\rho(\rho^*)\).\(^11\) In our case, we impose a correlation matrix \(\tau\) to the multivariate normal in Equation 14 and simulate a sample from the multivariate "Mixed" distribution. Then we

\(^8\)We have used two categories of tests: graphical tests and statistical tests. The first category includes QQ-plots, and the study of the empirical density function and empirical distribution function with respect to theoretical ones. The second category includes Jaque Bera normality tests, Chi-square tests, Anderson Darling tests, Kolmogorov-Smirnov tests and Cramer von Mises tests (see DAgostino and Stephens [6]).


\(^10\)Note that it is not a multivariate Student-t distribution, because each \(t\) has different degrees of freedom. Moreover the covariance between marginal random variables is no more proportional to the covariance matrix used in equation 21 to simulate from that multivariate distribution.

\(^11\)\(\rho^*\) used is indicated in (21).
vary τ until the correlation matrix ρ estimated on simulated data is equal to the desired correlation matrix ρ∗. This calibration method has been recently used also in Palmintesta and Provasi [19] in order to fit the parameters of the Koehler-Symanowski distributions on real data. They minimize the distance between the correlation matrix simulate from the multivariate Koehler-Symanowski distribution and correlation matrix estimated on real data.

We now examine the effects on asset allocation strategies. In particular, we can note that results obtained combining Student-t with different degrees of freedom (see Appendix D) are intermediate between those obtained using a multivariate Student-t with 7 degrees of freedom and with 10 degrees of freedom. This aspect is very important in a risk management framework because the use of different marginal distributions gives more information than previous analysis. This means that financial manager can invest with higher precision in the estimates of optimal strategies and of expected returns. Furthermore, it is important to note that for different combinations (for example t(15), t(3) and t(9)), where the leptokurtosis degree is very different, the loss of information could be very high if we choose a distribution with identically distributed marginal distributions.

Following the same techniques used in previous sections, we now concentrate our analysis on the study of effective risk associated with the investment if the financial manager uses a distribution Pm ≠ Pt.

We have seen that we can quantify these effects in at least two different ways: first, we can use the strategy xm while keeping the required shortfall return rlow constant and compute the actual shortfall probability α∗ a under the true probability measure Pt. Alternatively, we can use the strategy xm while keeping the required shortfall probability constant and compute the corresponding shortfall return r∗low.

Figure 6 indicates the level of α∗ if the manager uses the distribution t(νm) and the data follow the Mixed distribution. This analysis has been made, using α = 5% and α = 1%.

For α = 5% , if the investment manager uses a distribution that has thinner tails than reality, then the manager is conservative, in the sense that the shortfall constraint in (2) is not binding, while if the investment manager uses a distribution that has fatter tails

\footnote{We have set r\text{low} = 0\%.}
than reality, the shortfall constraint is violated.

As we know, if we consider the case \( \alpha = 1\% \), the results are reversed. If a thin-tailed distribution is assumed for the asset returns, e.g., the one based on normality, and if reality is leptokurtic, then the shortfall constraint is violated. Moreover, if \( \nu_m < \nu_t \), the shortfall constraint is not binding.

It is interesting to note the relationship that exists between Mixed distribution, \( t(7) \) and \( t(10) \). If we use a shortfall probability \( \alpha = 5\% \) in the model, then the values of true shortfall probability \( \alpha^*(\nu_m) \) are not only intermediate to those obtained using the other two distributions, but we can demonstrate that they are very close to what we could have by generating data from a \( t(7) \), for every level of \( \nu_m \). If we use a shortfall probability \( \alpha = 1\% \) in the model, when the data come from "Mixed" distribution, the behaviour of \( \alpha^*(\nu_m) \) seems initially intermediate for \( \nu_m = 3 \), and very close to \( t(7) \) curve, for higher degrees of freedom. For example, if the manager uses a distribution \( t(3) \), when the data follow a Mixed distribution, we obtain \( \alpha^* = 0.75\% \), while if the manager uses a distribution with thinner tails, the true shortfall probability for Mixed distribution tends to the true shortfall probability for \( t(7) \) one. It is important to consider that the interval in which the curve oscillates is very small, and for this reason every variation seems imperceptible.

The second analysis we can do is to study the differences \( r_{low} - r_{low}^* \) in basis points, when data come from Mixed distribution. Results obtained are indicated in Table 3.

<table>
<thead>
<tr>
<th>( \nu_m )</th>
<th>( r_{low} = 0% )</th>
<th>( r_{low} = -5% )</th>
<th>( r_{low} = -10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-253</td>
<td>-157</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>-37</td>
<td>-59</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>-4</td>
<td>-3</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>23</td>
<td>21</td>
<td>-</td>
</tr>
<tr>
<td>( \infty )</td>
<td>27</td>
<td>45</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \nu_m )</th>
<th>( r_{low} = 0% )</th>
<th>( r_{low} = -5% )</th>
<th>( r_{low} = -10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>43</td>
<td>47</td>
<td>71</td>
</tr>
<tr>
<td>5</td>
<td>33</td>
<td>39</td>
<td>70</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>10</td>
<td>-16</td>
<td>-39</td>
<td>-67</td>
</tr>
<tr>
<td>( \infty )</td>
<td>-62</td>
<td>-137</td>
<td>-198</td>
</tr>
</tbody>
</table>

For \( \alpha = 5\% \), if the investment manager uses a distribution with tails heavier than reality, the loss could exceed \( r_{low} \). For \( \alpha = 1\% \), the loss could exceed \( r_{low} \) only if the investment manager uses a distribution with tails thinner than reality. Furthermore, for \( \alpha = 1\% \) and \( r_{low} = -5\% \), the impact of using the normal distribution to explain the empirical distribution behaviour if reality is "Mixed", is quite substantial. In fact the manager is willing to take losses exceeding 5\% of 137 basis points, i.e. an increase of over 30\%. It is interesting to note that for \( \nu_m = 7 \), the difference \( r_{low} - r_{low}^* \) is very small and for \( \alpha = 5\% \) it tends to zero, decreasing the \( r_{low} \) level.

For \( \alpha = 1\% \), the difference becomes smaller with the reduction of losses. From this study we can conclude that for the case analysed, a multivariate Student-\( t \) distribution with 7 degrees of freedom seems a good approximation of the Mixed distribution. For the financial manager it could be simpler to use a multivariate \( t(7) \) with identically distributed marginal distributions, without excessive loss of information on the true risk. However, if this result held for the specified combination used \( t(30), t(7), t(8), \) in general we can not extend our conclusions for all the combinations of marginal distributions. This means
that before any application, it is wise for the financial manager to analyse whether an approximation of the mixed distribution can cause losses of information and thus undesired effects for risk management.

8 Conclusion

We propose a Monte Carlo simulation approach to investigate the effects of heterogeneity, asymmetry and fat tails on optimal asset allocation problems with a shortfall constraint. Financial markets usually show a "non-normal" and "non-homogeneous" behaviour, since the returns distributions often exhibit heavy tails and asymmetry with different intensity in each asset class. In this work we choose multivariate skewed Student-\(t\) distributions with heterogeneous-components to show the salient effects of heterogeneity, asymmetry and fat tails in a financial decision context. We argue that the effects of specification errors on the optimal asset allocation depend on the shortfall probability level. We find a critical level of shortfall probability and exhibit how it varies as a function of the degrees of leptokurtosis and skewness of the true return distribution.

We focus on the problem of misspecification of the degrees of kurtosis in a symmetric case. This analysis is not very realistic, but has allowed us to discover the true risk for the financial manager when the behaviour of empirical distribution is incorrectly estimated.

Finally we have deal with the heterogeneity assumption and study the effect on the downside risk of assuming homogeneous-component multivariate Student-\(t\), while the true distribution has heterogeneous components.

For each of these analyses, we may conclude that a correct assessment of the fattailedness of asset returns is important for the determination of optimal asset allocation. If asset allocations are based on the normal distribution, the resulting allocation may be either inefficient or unfeasible if reality is non-normal. Both effects can be quite substantial. Then, it appears that the shortfall probability set by the investment manager plays a crucial role for the nature of the effect of leptokurtic asset returns. If the shortfall probability is set sufficiently high, using normal scenarios for the leptokurtic asset return leads to overly prudent and therefore inefficient asset allocations. If the shortfall probability is sufficiently small, however, the use of normal scenarios leads to unfeasible strategies if reality is fat-tailed. This second result implies that the actual VaR of a given portfolio may be underestimated if the tail behaviour of asset returns is not captured adequately.

In studying the heterogenous-component distribution we have also shown by simulation that another important aspect in financial analysis is the correct specification of the marginal return distributions. The assumption of Student-\(t\) distribution with misspecified degrees of freedom produces effective shortfall probability and effective shortfall return which differ from the desired ones. However these errors have known upper and lower bounds. Even if this result strictly depends on our parameter setting, we stress again that the use of stochastic simulation as suggested in this work, is a very effective instrument to analyse the sensitivity of optimal asset allocation to the parameters of the problem. The results are therefore valuable to investors who require a qualitative assessment of the reliability and sensitivity of their adopted investment strategies in case their models are potentially misspecified.
A. Moments of the skewed Student-\( t \) distribution.

Let us first consider a vector \( r \) of independent standard skewed Student-\( t \). The \( i \)-th component (denoted with \( \theta(\nu, \gamma) \)) of the mean vector is

\[
\theta(\nu_i, \gamma_i) = \mathbb{E}(r_i)
\]

\[
= \frac{2}{\gamma + \frac{1}{\gamma}} \left\{ \int_{-\infty}^{0} r_i h_{\nu} \left( \frac{r_i}{\gamma} \right) dr_i + \int_{0}^{+\infty} r_i h_{\nu} \left( r_i \gamma \right) dr_i \right\}
\]

\[
= \frac{2\gamma}{\gamma^2 + 1} \left\{ \int_{-\infty}^{0} \frac{\gamma^2 z_i^2 h_{\nu}(z_i) dz_i}{\gamma^2} + \int_{0}^{+\infty} \frac{\gamma^2 z_i^2 h_{\nu}(z_i) dz_i}{\gamma^2} \right\}
\]

\[
= \frac{\gamma^2 - 1}{\gamma} \int_{0}^{+\infty} 2z \left( \int_{0}^{+\infty} \frac{1}{\sqrt{2\pi \lambda}} e^{-\frac{z^2}{2\lambda}} \frac{1}{\Gamma(\nu/2)} \left( \frac{\nu}{2} \right)^{\nu/2} e^{-\frac{\nu}{2} \lambda} \lambda^{-\nu/2 - 1} d\lambda \right) dz
\]

\[
= \frac{\gamma^2 - 1}{\gamma} \int_{0}^{+\infty} \left( \int_{0}^{+\infty} \frac{2\sqrt{\lambda} z e^{\frac{-z^2}{2\lambda}}}{\sqrt{2\pi \lambda}} \frac{1}{\Gamma(\nu/2)} \left( \frac{\nu}{2} \right)^{\nu/2} e^{-\frac{\nu}{2} \lambda} \lambda^{-\nu/2 - 1} d\lambda \right)
\]

\[
= \frac{\gamma^2 - 1}{\gamma} \int_{0}^{+\infty} \left( \Gamma((\nu - 1)/2) \left( \frac{\nu}{\pi} \right)^{1/2} \right)
\]

In lines four and five we use the normal scale-mixture representation of a Student-\( t \) and the Fubini’s theorem respectively.

In order to find the variance-covariance matrix, consider first

\[
\mathbb{E}(r_i^2) = \frac{2}{\gamma + \frac{1}{\gamma}} \left\{ \int_{-\infty}^{0} r_i^2 h_{\nu} \left( \frac{r_i}{\gamma} \right) dr_i + \int_{0}^{+\infty} r_i^2 h_{\nu} \left( r_i \gamma \right) dr_i \right\}
\]

\[
= \frac{2\gamma}{\gamma^2 + 1} \left\{ \int_{-\infty}^{0} \frac{\gamma^4 z_i^2 h_{\nu}(z_i) dz_i}{\gamma^2} + \int_{0}^{+\infty} \frac{\gamma^4 z_i^2 h_{\nu}(z_i) dz_i}{\gamma^2} \right\}
\]

\[
= \frac{2\gamma}{\gamma^2 + 1} \left( \frac{1}{\gamma^3} + \frac{1}{\gamma^3} \right) \int_{0}^{+\infty} z^2 h_{\nu}(z) dz
\]

\[
= \frac{\gamma^6 + 1}{\gamma^3 + \gamma^2 \nu - 2}
\]

The last two lines follow from the symmetry of the p.d.f. \( h_{\nu} \) of the Student-\( t \). Now, due to the independence between the components of the random vector, the out-of-diagonal components of the variance-covariance matrix are null. The variance-covariance matrix is

\( \Xi = \text{diag}\{\xi_1, \ldots, \xi_n\} \), where

\[
\xi_i(\nu_i, \gamma_i) = \mathbb{E}(r_i^2) - \mathbb{E}(r_i)^2
\]

\[
= \frac{\gamma^6 + 1}{\gamma_i^3 + \gamma_i^2 \nu_i - 2} - \frac{\gamma_i^4 - 2\gamma_i^2 + 1}{\gamma_i^2} \frac{\nu_i}{\pi} \left( \frac{\Gamma((\nu_i - 1)/2)}{\Gamma(\nu_i/2)} \right)^2
\]

\[
= \frac{\gamma_i^4 - \gamma_i^2 + 1}{\gamma_i^2} \left( \frac{\nu_i}{\nu_i - 2} - \frac{\nu_i}{\pi} \left( \frac{\Gamma((\nu_i - 1)/2)}{\Gamma(\nu_i/2)} \right)^2 \right) + \frac{\nu_i}{\pi} \left( \frac{\Gamma((\nu_i - 1)/2)}{\Gamma(\nu_i/2)} \right)^2.
\]
B. Fraction invested, varying the shortfall probability

![Graph showing fraction invested in Cash, Stocks, and Bonds varying the shortfall probability level.]

Figure 7: Fraction invested in Cash, Stocks and Bonds varying the shortfall probability level.

![Graph showing expected return varying the shortfall return (α = 1%, α = 5%).]

Figure 8: Expected return, varying the shortfall return (α = 1%, α = 5%).
C. Effects of the shortfall return level on the optimal allocation

Figure 9: Fraction invested in Cash, Stock and Bond for different distributions for $\alpha = 1\%$ and $\alpha = 5\%$. 
D. Results obtained form mixed distribution

Table 4: Fraction invested in each asset and portfolio expected return for the mixed distribution. Standard deviations are in parenthesis.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0.5%</th>
<th>1%</th>
<th>5%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CASH</strong></td>
<td><strong>STOCK</strong></td>
<td><strong>BONDS</strong></td>
<td><strong>Expected Return</strong></td>
<td></td>
</tr>
<tr>
<td>0%</td>
<td>83.878 (0.111)</td>
<td>78.665 (0.112)</td>
<td>49.254 (0.203)</td>
<td>7.813 (0.343)</td>
</tr>
<tr>
<td>10.902 (0.110)</td>
<td>21.26 (0.111)</td>
<td>47.64 (0.216)</td>
<td>91.09 (0.261)</td>
<td>STOCK</td>
</tr>
<tr>
<td>0.028 (0.010)</td>
<td>0.083 (0.024)</td>
<td>3.101 (0.015)</td>
<td>0.481 (0.026)</td>
<td>BONDS</td>
</tr>
<tr>
<td>8.442 (0.012)</td>
<td>8.980 (0.013)</td>
<td>11.825 (0.026)</td>
<td>16.182 (0.043)</td>
<td>Expected Return</td>
</tr>
<tr>
<td><strong>-3%</strong></td>
<td><strong>CASH</strong></td>
<td><strong>STOCK</strong></td>
<td><strong>BONDS</strong></td>
<td><strong>Expected Return</strong></td>
</tr>
<tr>
<td>72.302 (0.139)</td>
<td>64.982 (0.139)</td>
<td>25.277 (0.232)</td>
<td>0.000 (0.000)</td>
<td>CASH</td>
</tr>
<tr>
<td>27.179 (0.136)</td>
<td>33.776 (0.139)</td>
<td>67.362 (0.232)</td>
<td>1.000 (0.000)</td>
<td>STOCK</td>
</tr>
<tr>
<td>0.518 (0.021)</td>
<td>1.238 (0.006)</td>
<td>7.340 (0.094)</td>
<td>0.000 (0.000)</td>
<td>BONDS</td>
</tr>
<tr>
<td>9.012 (0.016)</td>
<td>10.313 (0.016)</td>
<td>14.190 (0.034)</td>
<td>17.000 (0.000)</td>
<td>Expected Return</td>
</tr>
<tr>
<td><strong>-5%</strong></td>
<td><strong>CASH</strong></td>
<td><strong>STOCK</strong></td>
<td><strong>BONDS</strong></td>
<td><strong>Expected Return</strong></td>
</tr>
<tr>
<td>65.245 (0.164)</td>
<td>56.596 (0.167)</td>
<td>8.516 (0.239)</td>
<td>0.000 (0.000)</td>
<td>CASH</td>
</tr>
<tr>
<td>33.485 (0.165)</td>
<td>41.261 (0.172)</td>
<td>83.76 (0.463)</td>
<td>1.000 (0.000)</td>
<td>STOCK</td>
</tr>
<tr>
<td>1.209 (0.021)</td>
<td>2.142 (0.013)</td>
<td>8.031 (0.224)</td>
<td>0.000 (0.000)</td>
<td>BONDS</td>
</tr>
<tr>
<td>10.306 (0.019)</td>
<td>11.127 (0.020)</td>
<td>16.760 (0.043)</td>
<td>17.000 (0.000)</td>
<td>Expected Return</td>
</tr>
<tr>
<td><strong>-7%</strong></td>
<td><strong>CASH</strong></td>
<td><strong>STOCK</strong></td>
<td><strong>BONDS</strong></td>
<td><strong>Expected Return</strong></td>
</tr>
<tr>
<td>57.928 (0.185)</td>
<td>48.348 (0.202)</td>
<td>0.144 (0.055)</td>
<td>0.000 (0.000)</td>
<td>CASH</td>
</tr>
<tr>
<td>39.164 (0.195)</td>
<td>48.593 (0.214)</td>
<td>99.778 (0.081)</td>
<td>1.000 (0.000)</td>
<td>STOCK</td>
</tr>
<tr>
<td>2.907 (0.011)</td>
<td>3.068 (0.017)</td>
<td>0.077 (0.027)</td>
<td>0.000 (0.000)</td>
<td>BONDS</td>
</tr>
<tr>
<td>10.395 (0.021)</td>
<td>11.917 (0.025)</td>
<td>10.392 (0.017)</td>
<td>17.000 (0.000)</td>
<td>Expected Return</td>
</tr>
<tr>
<td><strong>-10%</strong></td>
<td><strong>CASH</strong></td>
<td><strong>STOCK</strong></td>
<td><strong>BONDS</strong></td>
<td><strong>Expected Return</strong></td>
</tr>
<tr>
<td>47.797 (0.216)</td>
<td>35.883 (0.200)</td>
<td>0.000 (0.000)</td>
<td>0.000 (0.000)</td>
<td>CASH</td>
</tr>
<tr>
<td>48.208 (0.237)</td>
<td>60.088 (0.219)</td>
<td>1.000 (0.000)</td>
<td>1.000 (0.000)</td>
<td>STOCK</td>
</tr>
<tr>
<td>3.994 (0.030)</td>
<td>4.028 (0.023)</td>
<td>0.000 (0.000)</td>
<td>0.000 (0.000)</td>
<td>BONDS</td>
</tr>
<tr>
<td>11.938 (0.026)</td>
<td>13.151 (0.024)</td>
<td>17.000 (0.000)</td>
<td>17.000 (0.000)</td>
<td>Expected Return</td>
</tr>
</tbody>
</table>
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