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Department of Economics and Management

University of Brescia

Italy

C. Archetti, N. Bianchessi, S. Irnich
M.G. Speranza

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Via S. Faustino 74/b, 25122 Brescia – Italy
Tel 00390302988742 Fax 00390302899703
email: paqricercadem@eco.unibs.it

Formulations for an inventory routing problem

Claudia Archetti [(1)] *Nicola Bianchessi* [(1)] *Stefan Irnich* [(2)]

M. Grazia Speranza [(1)]

(1) *Department of Economics and Management, University of Brescia, Italy,*

Email: archetti@eco.unibs.it, bianche@eco.unibs.it, speranza@eco.unibs.it

(2) *Chair of Logistics Management, Johannes Gutenberg University Mainz, Germany,*

Email: irnich@uni-mainz.de

Abstract

In this paper we present and compare formulations for the Inventory Routing Problem (IRP) where the demand of customers has to be served, over a discrete time horizon, by capacitated vehicles starting and ending their routes at a depot. The objective of the IRP is the minimization of the sum of inventory and transportation costs. The formulations include known and new mathematical programming formulations. Valid inequalities are also presented. The formulations are tested on a large set of benchmark instances.

Keywords: Routing problems, Integer programming, Branch and cut, Supply chain management

1 Introduction

The interest in the integration of inventory and transportation planning has been growing constantly, enhanced by the advances in information technology. The body of literature available in the area is presented in the surveys (Bertazzi et al., 2008), (Cordeau et al., 2009) and (Andersson et al., 2010b). Several papers were motivated by applications, for example, in distribution of liquified natural gas through ships (Christiansen, 1999), (Andersson et al., 2010a), distribution of raw material to the paper industry (Dauzère-Pérès et al., 2007), distribution of food to supermarkets (Mercer and Tao, 1996), and propose and study a specific model. Despite the work done in the area, the mathematical formulations for the basic problems are not yet well established, contrary to the case of the more classical vehicle routing problems.

We consider the Inventory Routing Problem (IRP) where customers have to be served over a discrete time horizon by a fleet of capacitated vehicles starting and ending their routes at a depot. The problem is to decide in each time period how much to deliver to each customer and the routes of the vehicles in such a way that the sum of inventory and transportation costs is minimized. The single vehicle version of this problem was introduced in (Bertazzi et al., 2002) where a heuristic approach was proposed. A branch-and-cut algorithm and a matheuristic were presented in (Archetti et al., 2007) and (Archetti et al., 2012), respectively. A tighter formulation than the one proposed in (Archetti et al., 2007) was given in (Solyalı and Süral, 2011). Instances with up to 60 customers were solved to optimality over an horizon of three days, and up to 15 customers over 12 days within four hours.

The multi-vehicle version of the problem was modeled and solved heuristically in (Coelho et al., 2012), whereas in (Adulyasak et al., 2013b) and (Coelho and Laporte, 2013) exact approaches were proposed for more general problems.

The goal of this paper is to analyze in depth different formulations of the multi-vehicle problem, together with valid inequalities, and to systematically test their performance on a large set of benchmark instances. We consider and compare the known formulations and propose a new one. Subtour elimination constraints are required in only one of the formulations. Their effectiveness as valid inequalities is tested on the others. The computational experiments are run for both the case of limited and unlimited fleet.

The IRP is defined in Section 2. The mathematical programming formulations and valid inequalities are presented in Sections 3 and 4, respectively. The characteristics of the branch-and-cut algorithm used for the solution of the different tested formulations are described in Section 5. In Section 6 the tested instances are presented together with the computational results. Final conclusions are drawn in Section 7.

2 Problem definition

Given a time horizon of H time periods, the problem is defined over a complete undirected graph $G = (N, E)$, where node 0 is the depot and nodes in $N' = \{1, ..., n\}$ represent the customers. A per period unit inventory holding cost h_i and an initial inventory level I_{i0} are associated with each node $i \in N$. A maximum inventory level U_i is also associated with each customer $i \in N'$. Let T denote the set of time periods $\{1, \ldots, H\}$. It is assumed that, in each time period $t \in T$, r_{0t} units of a product are made available at the depot, and r_{it} units are consumed at customer $i \in N'$. Each edge $(i, j) \in E$ represents the possibility to travel between locations i and j at non-negative cost c_{ij}. A fleet of m identical vehicles with a capacity Q is available to provide the service. In each time period inventory costs are incurred at the depot and at the customers and, if some customers are replenished, transportation costs have to be paid, too. The Inventory Routing Problem (IRP) is to determine for each time period $t \in T$ the quantity to deliver to each customer $i \in N'$ and the routes visiting the customers served at time t at the minimum total cost. This includes the inventory costs at all nodes and the costs of the vehicle routes. Decisions have to be taken in such a way that no stock-out occurs at the customers, the availability at the depot of the quantity delivered is guaranteed, and the vehicle capacity constraints are satisfied. We consider two replenishment policies. The Order-Up-to level (OU) policy, according to which every visit to a customer brings its inventory to the maximum level, and the Maximum Level (ML) policy, where any quantity can be delivered to the customers, provided that the maximum level is not exceeded. The formulations will have to account also for the respective replenishment policy. In the following we use the notation $E(S) = \{(i, j) \in E | i \in S, j \in S\}$ and $\delta(S) = \{(i, j) \in E | (i \in S, j \notin S)$ or $(i \notin S, j \in S)\}$.

3 Formulations

In each time period of the horizon various activities take place at the depot: an amount of product is made available, a quantity is delivered, and the inventory level is changed. Similarly, at each retailer the quantity delivered is received, an amount is consumed, and the inventory level is changed. The sequence of the activities in the time period influences the mathematical programming formulations. Besides the assumption on the sequence of the activities in a time period, other assumptions from the literature have generated different formulations and some confusion. We try to clarify things here.

In (Archetti et al., 2007) the sequence of the activities in a time period is assumed to be different from the other papers cited above: ‘inventory level calculation-delivery-consumption’. This sequence implies that the inventory level at time t is the inventory level at time $t-1$ plus the amount delivered at time $t-1$ minus the amount consumed at time $t-1$.

In (Coelho et al., 2012), (Adulyasak et al., 2013b), (Coelho and Laporte, 2013) and (Archetti et al., 2012) it is assumed that, both at the depot and at the retailers, the inventory level calculation is the last activity. At each customer the sequence is ’delivery-consumption-inventory level calculation’. In formulae, this sequence implies that the inventory level at time t is the inventory level at time $t-1$ plus the amount delivered at time t minus the amount consumed at time t. This assumption is consistent with the literature on lot sizing (Pochet and Wolsey, 2006).

A further assumption in (Archetti et al., 2012), different from the other three cited papers, is that the quantity received by a customer is such that the inventory level, calculated at the end of the time period, may equal the maximum level. This implies that, before consumption, the amount of product may exceed the maximum level. However, the inventory level will never exceed the maximum level because it is calculated after consumption. In formulae, this latter assumption implies that the inventory level at time t cannot be higher than the maximum level. In the other three papers the inventory level at time t cannot be greater than the difference of the maximum level and the quantity delivered at t.

In order to be aligned with the literature on lot sizing we maintain the assumption on the sequence 'delivery-consumption-calculation of inventory level'. Moreover, following the most recent papers we assume that the inventory level at time t cannot be greater than the maximum level minus the quantity delivered at t.

The formulations described in the remaining of this section can be classified in two main groups according to the fact that indices are associated with the vehicles or not. When indices are associated with the vehicles we consider a vehicle index set $K = \{1, \ldots, m\}$.

3.1 Vehicle indexed formulations

These formulations will be denoted with $(\mathrm{k}-\mathrm{x}-\mathrm{y})$ in the following, where k stands for 'with vehicle index **k**', and 'x' and 'y' specify the type of formulation and the replenishment policy, respectively.

3.1.1 (Archetti et al., 2007) formulation extensions

The first model we present is an extension of the formulation proposed by Archetti et al. (2007) for the single vehicle version of the problem. Formulations similar to this model will be referred to as $(\cdot-\mathrm{A}-\cdot)$ in the following. We start with the formulation for the OU policy denoted by $(\mathrm{k}-\mathrm{A}-\mathrm{ou})$.

The model makes use of variables I_{it} to indicate the inventory level at node $i \in N$ at the end of time period $t \in T$. The binary variable z_{it}^k is equal to 1 if node $i \in N$ is visited at time period $t \in T$ by vehicle $k \in K$, the variable q_{it}^k represents the quantity delivered to customer $i \in N'$ at time period $t \in T$ by vehicle $k \in K$, and the variable y_{ij}^{kt} represents the number of times the edge $(i, j) \in E$ is traversed by vehicle $k \in K$ in time period $t \in T$. From the viewpoint of the key decisions to take, the $\mathbf{z}, \mathbf{q}$ and $\mathbf{y}$ variables indicate when to perform the deliveries to the customers, how much to deliver, and how to build the routes, respectively.

$$
\begin{aligned}
\min \quad & \sum_{t\in T} h_0 I_{0t} + \sum_{i\in N'}\sum_{t\in T} h_i I_{it} + \sum_{k\in K}\sum_{(i,j)\in E}\sum_{t\in T} c_{ij} y_{ij}^{kt} && \text{(1a)}\\
\text{s.t.} \quad & I_{0t} = I_{0,t-1} + r_{0t} - \sum_{k\in K}\sum_{i\in N'} q_{it}^k \qquad t\in T && \text{(1b)}\\
& I_{it} = I_{i,t-1} - r_{it} + \sum_{k\in K} q_{it}^k \qquad i\in N',\ t\in T && \text{(1c)}\\
& I_{it} \ge 0 \qquad i\in N,\ t\in T && \text{(1d)}\\
& \sum_{k\in K} q_{it}^k \ge U_i \sum_{k\in K} z_{it}^k - I_{it-1} \qquad i\in N',\ t\in T && \text{(1e)}\\
(\text{k}-\text{A}-\text{ou}) \quad & \sum_{k\in K} q_{it}^k \le U_i - I_{it-1} \qquad i\in N',\ t\in T && \text{(1f)}\\
& q_{it}^k \le U_i z_{it}^k \qquad i\in N',\ k\in K,\ t\in T && \text{(1g)}\\
& \sum_{i\in N'} q_{it}^k \le Q z_{0t}^k \qquad k\in K,\ t\in T && \text{(1h)}\\
& \sum_{k\in K} z_{it}^k \le 1 \qquad i\in N',\ t\in T && \text{(1i)}\\
& \sum_{j:(i,j)\in E} y_{ij}^{kt} = 2 z_{it}^k \qquad i\in N,\ k\in K,\ t\in T && \text{(1j)}\\
& \sum_{(i,j)\in E(S)} y_{ij}^{kt} \le \sum_{i\in S} z_{it}^k - z_{st}^k \quad S\subseteq N',\ s\in S,\ k\in K,\ t\in T && \text{(1k)}\\
& z_{it}^k \in \{0,1\} \qquad i\in N,\ k\in K,\ t\in T && \text{(1l)}\\
& q_{it}^k \ge 0 \qquad i\in N',\ k\in K,\ t\in T && \text{(1m)}\\
& y_{ij}^{kt} \in \{0,1\} \qquad \{i,j\}\in E,\ k\in K,\ t\in T && \text{(1n)}\\
& y_{0j}^{kt} \in \{0,1,2\} \qquad j\in N',\ k\in K,\ t\in T && \text{(1o)}
\end{aligned}
$$

The objective function (1a) calls for the minimization of the total operational cost, that is the sum of the inventory costs at the depot, the inventory costs at the customers, and the costs of the routes over the time horizon. Constraints (1b)-(1d) determine the evolution of the inventory level over time and force the absence of stock-out situations at the supplier and at the customers. Constraints (1e)-(1g) ensure the OU policy requirements imposing that, if a customer is visited, the quantity delivered is such that the maximum inventory level is reached. Constraints (1h) are the vehicle capacity constraints. Constraints (1i)-(1k) are the routing constraints. Constraints (1i) impose to visit each customer at most once in each time period, constraints (1j) are the degree constraints for each node and each vehicle in each time period, and (1k) are the subtour elimination constraints (SECs) for each vehicle route and each time period. Note that SECs (1k) are stronger than those with right-hand side equal to $|S|-1$.

If we remove constraints (1e) from model (k−A−ou), the resulting model (k−A−ml) applies for the ML policy.

3.1.2 (Solyalı and Süral, 2011) formulation extensions

For the single vehicle version of the problem and under the OU policy, Solyalı and Süral (2011) proposed a model stronger than the one presented in (Archetti et al., 2007). We describe their modeling approach using a different notation in order to better highlight the underlying network structure. We then present new formulations inspired by this approach, abbreviated with $(\cdot-\mathrm{S}-\cdot)$ in the following.

For any customer $i \in N'$, the basic idea of Solyalı and Süral (2011) is that the sequence of replenishment operations that occur at i has the structure of the well-known (single-item single-level) dynamic lot-sizing problem (Pochet and Wolsey, 2006). For this problem, it is known that a formulation with separate variables for setups, that is decisions on the periods of replenishment, and production/order quantities is weak, and that a path formulation over the time periods is stronger. Herein, binary variables indicate that customer $i \in N'$ is replenished at times τ and $t \in T$, but no replenishment takes place in between, at times $t' \in T$ with $\tau < t' < t$. Assuming the OU policy, on the basis of the initial inventory levels and demands it is possible to determine all possible combinations (τ, t) together with the delivery quantities.

The following notation is needed. We define an extended set of time periods $\mathscr{T} = T \cup \{0, H+1\} = \{0, 1, \dots, H, H+1\}$. We define also $r_{i,0} = r_{i,H+1} = 0$ for each $i \in N'$. Let $\mathscr{A}^i \subset \{(\tau,t) \in \mathscr{T} \times \mathscr{T} : \tau < t\}$ be the set of all pairs (τ, t) such that a replenishment takes place at τ and t without intermediate replenishment. The elements $(\tau,t) \in \mathscr{A}^i$ can be interpreted as arcs of the acyclic digraph $\mathscr{G}^i = (\mathscr{T}, \mathscr{A}^i)$. In the standard case for $\tau > 0$ and $t < H+1$, the fact that there is no intermediate replenishment between τ and t means that at time t customer i must receive the quantity $R_{i\tau,t-1}$, where $R_{i\tau t}$ is defined as $R_{i\tau t} := \sum_{t'=\tau}^{t} r_{it'}$. Therefore, the replenishment sequence is feasible, indicated by $(\tau,t) \in \mathscr{A}^i$, if and only if $R_{i\tau,t-1} \le U_i$.

The time period $t = H+1$ stands for the end of the time horizon. Consequently, an arc $(\tau, H+1) \in \mathscr{A}^i$ models the fact that the final replenishment at time τ is feasible, meaning that the quantity $R_{i\tau,H+1}$ consumed from τ to the end of the planning horizon does not exceed the maximum level U_i. Hence, $(\tau, H+1) \in \mathscr{A}^i$ if and only if $R_{i\tau,H+1} \le U_i$. Note that in all cases with $\tau > 0$ the associated quantity $R_{i\tau,t}$ replenished at time t equals the quantity consumed by i in the period between τ and $t-1$.

We now discuss the case where the initial inventory I_{i0} suffices to cover the demand of periods $1, \dots, t-1$.

The case with $\tau = 0$ is different due to the initial inventory level I_{i0}. An arc $(0,t) \in \mathscr{A}^i$ describes the fact that the first replenishment takes place at time t, which is feasible if and only if $I_{i0} \geq R_{i\tau,t-1}$. Note that we assume $(0,H+1) \notin \mathscr{A}^i$ because otherwise the initial inventory would suffice to cover the entire demand of customer i. In this case no visit to i would be needed. The quantity consumed by customer i is $R_{i1,t-1}$ (as before), but the quantity to replenish is $U_i - (I_{i0} - R_{i\tau,t-1})$. By definition of $\mathscr{A}^i$ this quantity $U_i - (I_{i0} - R_{i\tau,t-1})$ is non-negative.

We aggregate the quantities that are replenished at time t, when the last replenishment took place in τ, to the quantities $b_{i\tau t}$ defined by

$$b_{i\tau t} = \begin{cases} U_i - (I_{i0} - R_{i1,t-1}) & \text{for } (0,t) \in \mathscr{A}^i \\ R_{i\tau,t-1} & \text{for } (\tau,t) \in \mathscr{A}^i,\ \tau > 0. \end{cases} \tag{2}$$

Associating binary variables $w^k_{i\tau t}$ with arcs $(\tau,t) \in \mathscr{A}^i$, $i \in N'$, corresponding to consecutive replenishment operations at customer i, the following model can be formulated for the IRP:

$$\min \quad \sum_{t\in T} h_0 I_{0t} + \sum_{i\in N'}\sum_{t\in T} h_i I_{it} + \sum_{k\in K}\sum_{(i,j)\in E}\sum_{t\in T} c_{ij} y^{kt}_{ij} \tag{3a}$$

$$\text{s.t.} \quad I_{0t} = I_{0,t-1} + r_{0t} - \sum_{k\in K}\sum_{i\in N'}\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w^k_{i\tau t} \qquad t \in T \tag{3b}$$

$$I_{it} = I_{i,t-1} - r_{it} + \sum_{k\in K}\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w^k_{i\tau t} \quad i \in N',\ t \in T \tag{3c}$$

$$\sum_{k\in K}\sum_{t:(0,t)\in\mathscr{A}^i} w^k_{i0t} = 1 \qquad i \in N' \tag{3d}$$

$$(\text{k}-\text{S}-\text{ou}) \quad \sum_{k\in K}\sum_{\tau:(t,\tau)\in\mathscr{A}^i} w^k_{it\tau} - \sum_{k\in K}\sum_{\tau:(\tau,t)\in\mathscr{A}^i} w^k_{i\tau t} = 0 \quad t \in T,\ i \in N' \tag{3e}$$

$$\sum_{k\in K}\sum_{\tau:(\tau,H+1)\in\mathscr{A}^i} w^k_{i\tau,H+1} = 1 \qquad i \in N' \tag{3f}$$

$$\sum_{\tau:(\tau,t)\in\mathscr{A}^i} w^k_{i\tau t} = z^k_{it} \qquad i \in N',\ k \in K,\ t \in T \tag{3g}$$

$$\sum_{i\in N'}\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w^k_{i\tau t} \leq Q z^k_{0t} \qquad k \in K,\ t \in T \tag{3h}$$

$$(1i) - (1l), (1n), (1o) \tag{3i}$$

$$w^k_{i\tau t} \in \{0,1\} \qquad i \in N',\ (\tau,t) \in \mathscr{A}^i,\ k \in K \tag{3j}$$

Variables $w^k_{i\tau t}$ are indexed by k in order to correctly charge the corresponding quantity $b_{i\tau t}$ to the vehicles, as implied by the vehicle capacity constraints $(3h)$. Having the index k associated with variables $w^k_{i\tau t}$ does not imply that customer i is replenished by the same vehicle. The replenishment is however ensured by the replenishment constraints (3d)-(3f), modeling for each customer $i \in N'$ a path in the network $\mathscr{G}^i$. Actually, for each node of the directed graph $\mathscr{G}^i$ the corresponding constraint is modeled aggregating the ingoing and outgoing flows over all the vehicles.

The objective function (3a) is the same as (1a). Constraints (3b)-(3c) have the same meaning of (1b)-(1c). In particular here the quantity delivered to customer $i \in N'$ at time $t \in T$ is given by $\sum_{k \in K} \sum_{\tau:(\tau,t) \in \mathscr{A}^i} b_{i\tau t} w^k_{i\tau t}$. The construction of graph $\mathscr{G}^i = (\mathscr{T}, \mathscr{A}^i)$ guarantees that no stock-out occurs at customer $i \in N'$. Constraints (3g) are consistency constraints between the **w** and the **z** variables. Routing constraints are stated as in (k−A−ou).

It is interesting to note that, in case of the ML policy, the acyclic digraph $\mathscr{G}^i = (\mathscr{T}, \mathscr{A}^i)$ describes for customer $i \in N'$ a superset of all the feasible replenishment sequences, and gives useful information on the timing of the deliveries. We tried to exploit this information defining a formulation for the problem which makes use of the **w** variables also when the ML policy is assumed. The new formulation extends (k−A−ml) by including the path constraints (4c)-(4e): For each customer $i \in N'$, a feasible replenishment sequence is a path in $\mathscr{G}^i$. (4f) are consistency constraints between the **w** and the **z** variables. The index k is no more associated with variables **w** because here they are needed only to sequence the replenishment operations. The replenishment quantities do not depend on the timing of the deliveries and are directly modeled by variables **q** as in (k−A−ml).

$$\min \quad \sum_{t \in T} h_0 I_{0t} + \sum_{i \in N'} \sum_{t \in T} h_i I_{it} + \sum_{k \in K} \sum_{(i,j) \in E} \sum_{t \in T} c_{ij} y^{kt}_{ij} \tag{4a}$$

(k−S−ml)

$$\text{s.t.} \quad (1b) - (1d), (1f) - (1o) \tag{4b}$$

$$\sum_{t:(0,t) \in \mathscr{A}^i} w_{i0t} = 1 \qquad i \in N' \tag{4c}$$

$$\sum_{\tau:(t,\tau) \in \mathscr{A}^i} w_{it\tau} - \sum_{\tau:(\tau,t) \in \mathscr{A}^i} w_{i\tau t} = 0 \qquad t \in T,\ i \in N' \tag{4d}$$

$$\sum_{\tau:(\tau,H+1) \in \mathscr{A}^i} w_{i\tau,H+1} = 1 \qquad i \in N' \tag{4e}$$

$$\sum_{\tau:(\tau,t) \in \mathscr{A}^i} w_{i\tau t} = \sum_{k \in K} z^k_{it} \qquad i \in N',\ t \in T \tag{4f}$$

$$w_{i\tau t} \in \{0,1\} \qquad i \in N',\ (\tau,t) \in \mathscr{A}^i \tag{4g}$$

3.2 Flow formulations

In this second group of formulations, vehicles are not associated with indices. Thus, they will be denoted with (nk−x−y) in the following, where nk stands for '**n**o vehicle index **k**'. The variables **q**, **z** and **y** are used here without the index k but otherwise identical meaning.

The name given to this group of formulations is due to the use of continuous variables modeling the flow of the load of the vehicles traversing the links of the network. The formulations make use of new variables **l** and **x**. l_{ij}^t represents the load of a vehicle traveling from $i \in N$ to $j \in N$, at time period $t \in T$. Since the **l** variables' definition implies knowing the orientation of the edges traversed, binary variables **x** are introduced: x_{ij}^t is equal to 1 if edge $(i, j) \in E$ is traversed from node i to node j by a vehicle at time period $t \in T$ and 0 otherwise. Let $A = \{(i, j), (j, i) : \{i, j\} \in E\}$ be the arc set corresponding to edge set E of the graph representing the logistic network. x_{ij}^t is defined for each arc $(i, j) \in A$ and each time period $t \in T$.

3.2.1 Basic formulation

The first formulation models the problem for the of OU policy.

$$
\begin{aligned}
\min \quad & \sum_{t\in T} h_0 I_{0t} + \sum_{i\in N'}\sum_{t\in T} h_i I_{it} + \sum_{(i,j)\in E}\sum_{t\in T} c_{ij} y^t_{ij} && \text{(5a)}\\
\text{s.t.} \quad & I_{0t} = I_{0,t-1} + r_{0t} - \sum_{i\in N'} q_{it} \qquad t\in T && \text{(5b)}\\
& I_{it} = I_{i,t-1} - r_{it} + q_{it} \qquad i\in N',\ t\in T && \text{(5c)}\\
& I_{it} \geq 0 \qquad i\in N,\ t\in T && \text{(5d)}\\
& q_{it} \geq U_i z_{it} - I_{it-1} \qquad i\in N',\ t\in T && \text{(5e)}\\
& q_{it} \leq U_i - I_{it-1} \qquad i\in N',\ t\in T && \text{(5f)}\\
& q_{it} \leq U_i z_{it} \qquad i\in N',\ t\in T && \text{(5g)}\\
& \sum_{j:(i,j)\in E} y^t_{ij} = 2 z_{it} \qquad i\in N',\ t\in T && \text{(5h)}\\
(\text{nk}-\text{A}-\text{ou}) \quad & \sum_{j|(i,j)\in A} l^t_{ij} - \sum_{j|(j,i)\in A} l^t_{ji} = \begin{cases} -q_{it} & \text{if } i\in N' \\ \sum_{i\in N'} q_{it} & \text{if } i = 0 \end{cases} \quad i\in N,\ t\in T && \text{(5i)}\\
& l^t_{ij} \leq Q x^t_{ij} \qquad (i,j)\in A,\ t\in T && \text{(5j)}\\
& \sum_{j|(j,i)\in A} x^t_{ji} - \sum_{j|(i,j)\in A} x^t_{ij} = 0 \quad i\in N,\ t\in T && \text{(5k)}\\
& x^t_{ij} + x^t_{ji} = y^t_{ij} \qquad (i,j)\in E,\ t\in T && \text{(5l)}\\
& \sum_{j:(0,j)\in A} x^t_{0j} \leq m \qquad t\in T && \text{(5m)}\\
& q_{it} \geq 0 \qquad i\in N',\ t\in T && \text{(5n)}\\
& z_{it} \in \{0,1\} \qquad i\in N,\ t\in T && \text{(5o)}\\
& y^t_{ij} \in \{0,1\} \qquad (i,j)\in E,\ t\in T && \text{(5p)}\\
& y^t_{0j} \in \{0,1,2\} \qquad j\in N',\ t\in T && \text{(5q)}\\
& x^t_{ij} \in \{0,1\},\ l^t_{ij} \geq 0 \qquad (i,j)\in A,\ t\in T && \text{(5r)}
\end{aligned}
$$

The objective function (5a) and the constraints (5b)-(5f) on the feasibility of the inventory levels and the OU policy requirements are similar to (1a) and (1b)-(1g), respectively. In contrast to formulation (k−A−ou), here the single visit to each customer in each time period is directly implied by the binary variables **z**. Constraints (5h)-(5k) are the routing and vehicle capacity constraints. Constraints (5h) are the degree constraints for each customer node of the graph in each time period. They ensure consistency between the values of the **y** and **z** variables. (5i) are flow conservation constraints for each node of the graph in each time period. Constraints (5k) are still degree constraints, but now they are defined for each node of the graph and in terms of **x** variables. These constraints ensure connectivity for each route according to the orientation of the traversed edges, whereas constraints (5i) impose each route to be elementary. Moreover, constraints (5k) allow us to state constraints (5j) which link in a consistent way the values of the **l** and **x** variables, and impose the capacity constraints along each arc. Constraints (5l) are consistency constraints between the **x** and the **y** variables. Finally, constraints (5m) impose an upper bound on the number of vehicles available in each time period.

If we remove constraints (5e) from model $(\text{nk}-\text{A}-\text{ou})$, the resulting model $(\text{nk}-\text{A}-\text{ml})$ applies for the ML policy.

3.2.2 Extended formulation

Also here, the use of the **w** variables allows us to define two new formulations starting from $(\text{nk}-\text{A}-\text{ou})$ and $(\text{nk}-\text{A}-\text{ml})$. Again, in case of the OU policy the **w** variables describe the feasible replenishment sequences. The quantity to deliver at each visit is known given the time of the last replenishment.

Reformulating (nk−A−ou) by making use of the **w** variables results in the following model:

$$\min \quad \sum_{t\in T} h_0 I_{0t} + \sum_{i\in N'}\sum_{t\in T} h_i I_{it} + \sum_{(i,j)\in E}\sum_{t\in T} c_{ij} y_{ij}^t \tag{6a}$$

$$\text{s.t.} \quad I_{0t} = I_{0,t-1} + r_{0t} - \sum_{i\in N'}\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w_{i\tau t} \qquad t\in T \tag{6b}$$

$$I_{it} = I_{i,t-1} - r_{it} + \sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w_{i\tau t} \qquad i\in N',\ t\in T \tag{6c}$$

$$(5h), (5j)-(5m) \tag{6d}$$

$$\text{(nk−S−ou)} \quad \sum_{j|(i,j)\in A} l_{ij}^t - \sum_{j|(j,i)\in A} l_{ji}^t = \begin{cases} -\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w_{i\tau t} & i\in N' \\ \sum_{i\in N'}\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w_{i\tau t} & i=0 \end{cases} \quad i\in N,\ t\in T \tag{6e}$$

$$\sum_{t:(0,t)\in\mathscr{A}^s} w_{i0t} = 1 \qquad i\in N' \tag{6f}$$

$$\sum_{\tau:(t,\tau)\in\mathscr{A}^i} w_{it\tau} - \sum_{\tau:(\tau,t)\in\mathscr{A}^i} w_{i\tau t} = 0 \qquad t\in T,\ i\in N' \tag{6g}$$

$$\sum_{\tau:(\tau,H+1)\in\mathscr{A}^i} w_{i\tau,H+1} = 1 \qquad i\in N' \tag{6h}$$

$$\sum_{\tau:(\tau,t)\in\mathscr{A}^i} w_{i\tau t} = z_{it} \qquad i\in N',\ t\in T \tag{6i}$$

$$w_{i\tau t} \ge 0 \qquad i\in N',\ (\tau,t)\in\mathscr{A}^i \tag{6j}$$

$$(5o)-(5r) \tag{6k}$$

The objective function is the same as in the previous formulations. Constraints (6b)-(6c) are equivalent to (5b)-(5d), given that, here, the quantity delivered to customer $i\in N'$ at time $t\in T$ is equal to $\sum_{\tau:(\tau,t)\in\mathscr{A}^i} b_{i\tau t} w_{i\tau t}$, and constraints ensuring that no stock-out occurs at the customers are no more necessary. Routing and vehicle capacity constraints are stated as in (nk−A−ou) with the exception of (5i) which are now formulated as in (6e). Consistency constraints (5l) and the constraints on the fleet size (5m) are kept in the formulation as defined in (nk−A−ou). Finally, path constraints (6f)-(6h), describing for each customer $i\in N'$ the feasible replenishment sequences, are inserted together with consistency constraints (6i).

For the ML policy, the **w** variables can be used only to model information concerning the timing of the deliveries. As done for extending (k−A−ml) to (k−S−ml), (nk−S−ml) extends (nk−A−ml) including the path constraints (7c)-(7e), which define for each customer $i\in N'$ the paths network represented by $\mathscr{G}^i$, and the consistency constraints (7f) between the **w** and the **z** variables.

$$
\begin{aligned}
\text{min} \quad & \sum_{t \in T} h_0 I_{0t} + \sum_{i \in N'} \sum_{t \in T} h_i I_{it} + \sum_{(i,j) \in E} \sum_{t \in T} c_{ij} y_{ij}^t && \text{(7a)} \\
\text{s.t.} \quad & (5b) - (5d),\ (5f) - (5r) && \text{(7b)} \\
& \sum_{t:(0,t) \in \mathscr{A}^s} w_{i0t} = 1 \qquad i \in N' && \text{(7c)} \\
(\text{nk}-\text{S}-\text{ml}) \quad & \sum_{\tau:(t,\tau) \in \mathscr{A}^i} w_{it\tau} - \sum_{\tau:(\tau,t) \in \mathscr{A}^i} w_{i\tau t} = 0 \qquad i \in N',\ t \in T && \text{(7d)} \\
& \sum_{\tau:(\tau,H+1) \in \mathscr{A}^i} w_{i\tau,H+1} = 1 \qquad i \in N' && \text{(7e)} \\
& \sum_{\tau:(\tau,t) \in \mathscr{A}^i} w_{i\tau t} = z_{it} \qquad i \in N',\ t \in T && \text{(7f)} \\
& w_{i\tau t} \geq 0 \qquad i \in N',\ (\tau,t) \in \mathscr{A}^i && \text{(7g)}
\end{aligned}
$$

3.3 Known formulations

In (Adulyasak et al., 2013b) four formulations for the problem under study were proposed, classified in vehicle index and non-vehicle index formulations. The vehicle index formulations are named $F(OU)|k$ and $F(ML)|k$ and model the problem when the OU and the ML policy is applied, respectively. These formulations, in their basic version, are equivalent to (k−S−ou) and (k−A−ml), respectively. The non-vehicle index formulations are named $F(OU)|nk$ and $F(ML)|nk$ and are derived from $F(OU)|k$ and $F(ML)|k$, respectively, without making use of the vehicle index and modeling the capacity and routing constraints as constraints similar to the generalized fractional subtour elimination constraints (GFSECs) for the VRPs. These latter formulations have the advantage of using fewer variables than the vehicle index formulations (the number of variables decreases proportionally according to the fleet size), but still have an exponential size in terms of number of constraints. On the contrary, the flow formulations proposed in Section 3.2 keep the number of variables small while being polynomial in terms of constraints number. In (Coelho and Laporte, 2013) only vehicle index formulations are proposed. These models are similar to (k−A−ou) and (k−A−ml). In particular here, for each time period, SECs are replaced by the three-index version of the Miller, Tucker, and Zemlin (1960) constraints. This gives rise to polynomial size formulations with a large number of variables.

4 Valid inequalities

4.1 Inherited valid inequalities

Some of the inequalities we considered are extensions of the valid inequalities proposed by Archetti et al. (2007) for the single vehicle version of the problem.

The first inequalities strengthen the replenishment policy constraints. The inequalities state that if between time periods $t-\tau$ and t a customer is not visited, the inventory level at $t-\tau$ must be sufficient to cover the consumption between the two time periods. Distinguishing between vehicle index and flow formulations, the inequalities (SC1) are formulated as

$$I_{it-\tau} \geq \left(1 - \sum_{k\in K}\sum_{t'=t-\tau+1}^{t} z_{it'}^{k}\right)\left(\sum_{t'=t-\tau+1}^{t} r_{it'}\right) \; i \in N',\; t \in T,\; \tau = 0,\ldots,t-1 \quad (8a)$$

and

$$I_{it-\tau} \geq \left(1 - \sum_{t'=t-\tau+1}^{t} z_{it'}\right)\left(\sum_{t'=t-\tau+1}^{t} r_{it'}\right) \quad i \in N',\; t \in T,\; \tau = 0,\ldots,t-1, \quad (8b)$$

respectively.

The second group of inequalities are consistency constraints between the **z** and the **y** variables, strengthening the routing part of the models.

(SC2-a)

$$z_{it}^{k} \leq z_{0t}^{k} \quad i \in N',\; t \in T,\; k \in K \quad (9a)$$

$$y_{ij}^{kt} \leq z_{it}^{k}, y_{ij}^{kt} \leq z_{jt}^{k} \quad i \in N', j \in N',\; i < j,\; t \in T,\; k \in K \quad (9b)$$

$$y_{j0}^{kt} \leq 2z_{0t}^{k}, y_{j0}^{kt} \leq 2z_{jt}^{k} \quad i \in N', j \in N',\; i < j,\; t \in T,\; k \in K \quad (9c)$$

$$y_{j0}^{t} \leq 2z_{jt} \quad j \in N',\; t \in T \quad (9d)$$

$$y_{ij}^{t} \leq z_{it}, y_{ij}^{t} \leq z_{jt} \quad i \in N', j \in N',\; i < j,\; t \in T \quad (9e)$$

Inequalities (9a)–(9c) are applied to strengthen vehicle index models, whereas (9d) and (9e) are applied to flow models.

4.2 Additional valid inequalities

In order to strengthen the vehicle index formulations we include symmetry breaking constraints (SBCs). These constraints are needed to avoid duplication of solutions that look different due to a different numbering of the vehicles. We consider two different forms of SBCs. The first SBCs, called here (SBCs−a), are the simplest ones and state that in time period $t \in T$ customer 1 must be visited by vehicle 1, customer 2 must be visited by vehicle 1 or 2 and so forth. The formulation of these constraints is

$$\text{(SBCs}-\text{a)} \qquad \sum_{k=i+1}^{m} z_{it}^{k} = 0 \quad i \in N',\ t \in T. \tag{10}$$

We then consider the (SBCs−b) that turn out to be the best among those tested by Adulyasak et al. (2013b):

$$z_0^{kt} \geq z_0^{k+1,t} \quad 1 \leq k \leq m-1,\ t \in T \tag{11a}$$

$$\text{(SBCs}-\text{b)} \quad \sum_{i=1}^{j} 2^{(j-i)} z_i^{kt} \geq \sum_{i=1}^{j} 2^{(j-i)} z_i^{k+1,t} \quad j \in N',\ 1 \leq k \leq m-1,\ t \in T. \tag{11b}$$

Constraints (11a) state that at time t vehicle $k+1$ cannot be used if vehicle k is not used. Thus, these constraints set an order on the use of the vehicles. Additionally, constraints (11b) ensure that, if an additional vehicle k is used, then the customer with smallest index not served by vehicles $1, \ldots, k-1$ is assigned to vehicle k.

Then, for all models we consider fractional capacity-cut constraints (FCCCs) which are similar to the classical capacity-cut constraints for the VRPs. The main difference is that here we cannot round up the right-hand side term, since the delivered quantities are described by variables. FCCCs are equivalent to the constraints similar to GFSECs proposed by Adulyasak et al. (2013b). The formulations of FCCCs for the different models are as follows:

$$Q \sum_{k \in K} \sum_{(i,j) \in \delta(S)} y_{ij}^{kt} \geq 2 \sum_{k \in K} \sum_{i \in S} q_{it}^{k} \ S \subseteq N',\ t \in T \tag{12a}$$

$$Q \sum_{k \in K} \sum_{(i,j) \in \delta(S)} y_{ij}^{kt} \geq 2 \sum_{k \in K} \sum_{i \in S} \sum_{\tau:(\tau,t) \in \mathscr{A}^i} b_{i\tau t} w_{i\tau t}^{k} \ S \subseteq N',\ t \in T \tag{12b}$$

$$\text{(FCCCs)} \quad Q \sum_{(i,j) \in \delta(S)} y_{ij}^{t} \geq 2 \sum_{i \in S} q_{it} \ S \subseteq N',\ t \in T \tag{12c}$$

$$Q \sum_{(i,j) \in \delta(S)} y_{ij}^{t} \geq 2 \sum_{i \in S} \sum_{\tau:(\tau,t) \in \mathscr{A}^i} b_{i\tau t} w_{i\tau t} \ S \subseteq N',\ t \in T \tag{12d}$$

Constraints (12a) apply to (k−A−ou), (k−A−ml) and (k−S−ml), whereas constraints (12b) apply to (k−S−ou). Load flow models (nk−A−ou), (nk−A−ml) and (nk−S−ml) are strengthened by constraints (12c). Finally, constraints (12d) apply to (nk−S−ou).

For the flow models, we consider also the possibility to strengthen the formulation by means of SECs formulated as follows:

$$\text{(SECs)} \qquad \sum_{(i,j)\in E(S)} y^t_{ij} \leq \sum_{i\in S} z_{it} - z_{st} \quad S \subseteq N',\ t \in T,\ s \in S \tag{13}$$

5 Branch-and-cut algorithms

Table I describes the way the branch-and-cut algorithms are defined according to the different formulations. Inequalities inherited from Archetti et al. (2007) are statically added to all formulations at the root node of the branch-and-bound tree. Also the SBCs are added at the root node of the vehicle index formulations.

On the other hand, SECs and FCCCs are dynamically inserted when violated. Checks for violation are done at each node of the branch-and-bound tree. It must be noticed that, to the sake of correctness, SECs are required in case of vehicle index models. Violated SECs are identified by means of Padberg and Rinaldi's algorithm described in (Padberg and Rinaldi, 1991). The violated subtour elimination constraints are introduced with $s = argmax_{j\in S}\{z^k_{jt}\}$ and $s = argmax_{j\in S}\{z_{jt}\}$ for the k and nk formulations, respectively.

Violated FCCCs are heuristically identified by means of the extended and greedy shrinking heuristics proposed in (Ralphs et al., 2003). Given the support graph of the linear relaxation solution, the extended shrinking heuristic begins by considering each pair i, j of customer nodes of the support graph. If the flow on the cut corresponding to the customer nodes pair violates a fractional capacity-cut constraint, the inequality induced by set $S = \{i, j\}$ is inserted in the set of the violated inequalities. Then, an edge between two customer nodes associated with the largest flow is considered. The edge is then shrunk into a supernode by merging the two adjacent nodes and summing the flows associated with the merged edges. The process is repeated on the contracted graph. The greedy shrinking heuristic begins by considering a set S comprising the customer node i of the support graph characterized by the largest global flow on the edges incident to it. Then a customer node j which is associated with the largest flow coming from nodes in S is iteratively inserted in the set S. If the flow on the cut corresponding to S violates a fractional capacity constraint, the corresponding inequality is inserted among the inequalities violated by the linear relaxation solution.

When branching is required, all formulations give priority to the **z** variables except for formulations

Table I: Summary of branch-and-cut features

	Formulations							
	OU policy				ML policy			
	(k−A)	(k−S)	(nk−A)	(nk−S)	(k−A)	(k−S)	(nk−A)	(nk−S)
SC1	•		•		•	•	•	•
SC2-a	•	•	•	•	•	•	•	•
SBCs	•	•			•	•		
FCCCs	•	•	•	•	•	•	•	•
SECs	•	•	•	•	•	•	•	•

where **w** variables are used. In this case, priority is given to the **w** variables first and then to the **z** variables. The search tree is explored using the best bound strategy.

6 Computational results

Computational tests have been performed on instances proposed for the single vehicle case in (Archetti et al., 2007). Here, the capacity of the vehicles has been divided by the fleet size m, which is set to 2 and 3 for $n \leq 25$ and to 3 and 4 for $n > 25$, as in (Adulyasak et al., 2013b).

We performed a sequence of tests:

- We ran preliminary tests to evaluate the performance of the different formulations. For these tests, instances with up to 30 customers for $H = 3$ and up to 20 customers for $H = 6$ were considered. The results are shown in Section 6.1.

- On the basis of the preliminary tests, we chose the best formulation for both the ML and the OU policy. We then tested the effectiveness of FCCCs for all formulations and of SECs for flows formulations.

- We ran the final tests on instances with up to 50 customers for $H = 3$, and up to 30 customers for $H = 6$.

For the first three sets of tests, the maximum CPU time was set to 1800 seconds and each run made on a single thread. For the final tests, the maximum CPU time was set to two hours (7200s) and each run made on a single thread.

Computational tests were made on a Intel(R) Xeon(R) CPU E5520, 2.27 GHz, 12.0 GB RAM. Code was written in C++, compiled with MS Visual Studio 2010 Express Edition in release mode, and CPLEX 12.4 was used as an exact solver.

6.1 Preliminary tests

Results on preliminary tests are shown in Table II for the ML policy and in Table III for the OU policy. For both tables, n is the number of customers and H is the time horizon. Then for each formulation, the following information is provided: '# solved' is the number of instances solved to optimality within the time limit, '% ip gap' is the average optimality gap at the end of the computation while '%lb gap' gives the average gap between the lower bound at the end of the computation and the best upper bound found by all formulations. Each row gives the average results over 20 instances.

Table II: Preliminary tests: ML policy

		(k−A−ml)			(k−S−ml)			(nk−A−ml)			(nk−S−ml)		
n	H	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap
5	3	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00
10	3	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00
15	3	16	7.65	0.50	16	4.51	0.21	14	2.47	0.17	13	2.20	0.22
20	3	9	10.62	1.13	7	9.43	1.75	6	6.32	0.44	7	6.62	0.39
25	3	5	12.57	1.67	4	11.58	0.80	3	4.71	0.20	3	5.16	0.36
30	3	0	23.02	2.74	0	19.16	3.18	1	10.01	0.08	1	12.86	0.24
5	6	19	0.29	0.03	17	1.43	0.23	20	0.00	0.00	19	0.19	0.02
10	6	4	5.57	1.72	4	6.35	2.14	2	2.26	0.26	2	2.24	0.32
15	6	0	13.51	3.59	0	16.91	3.61	0	5.23	0.01	0	9.14	0.16
20	6	0	16.13	3.35	0	21.34	3.40	0	11.36	0.05	0	9.47	0.05

@

Table III: Preliminary tests: OU policy

		(k−A−ou)			(k−S−ou)			(nk−A−ou)			(nk−S−ou)		
n	H	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap
5	3	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00
10	3	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00
15	3	18	7.91	0.37	20	0.00	0.00	20	0.00	0.00	18	3.72	0.25
20	3	0	7.25	3.35	4	4.48	0.23	6	3.69	0.12	0	6.26	2.74
25	3	4	15.14	3.74	3	4.77	0.80	4	5.08	0.77	4	14.54	2.76
30	3	0	24.44	5.08	0	10.22	0.49	0	8.59	0.53	0	21.16	4.57
5	6	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00	20	0.00	0.00
10	6	2	10.15	3.45	0	3.22	0.90	2	4.00	0.26	4	7.36	1.45
15	6	0	19.85	4.17	0	14.54	0.15	0	9.25	0.36	0	20.65	3.69
20	6	0	25.69	4.40	0	28.54	0.54	0	15.85	0.28	0	23.13	4.13

On the basis of the preliminary results, we focused the following tests on the best vehicle index formulation and on the best flow formulation for each policy, namely on (k−A−ml) and (nk−A−ml) formulations for the ML policy, and (k−S−ou) and (nk−A−ou) for the OU policy.

6.2 Capacity cuts and subtour elimination constraints

In order to test the effectiveness of FCCCs and SECs, we chose instances with 25 and 30 customers for $H = 3$ and 15 and 20 customers for $H = 6$. Results are shown in Table IV for the ML policy and Table V for the OU policy. The columns have the same meaning as in Tables II and III.

Each row gives the average results over 20 instances. For formulations (nk−A−ml) and (nk−S−ou), we tested the performance of the formulations with and without FCCCs and SECs. For formulations (k−A−ml) and (k−A−ou), SECs are needed so that we made tests with and without FCCCs. Note that in the preliminary tests the formulations was run with FCCCs and SECs. However, the results are slightly different from the ones reported in Tables IV and V. This is not due to a different implementation, but simply due to a different processing behavior of the machine related to the fact that a higher number of processes were running in parallel (6 instead of 4). The results show that FCCCs and SECs are not beneficial for any of the policies. We also tried to insert FCCCs only at the root node but without any benefit.

Table IV: FCCCs and SECs: ML policy

n	H				(k−A−ml) + FCCCs						(k−A−ml)			(k−A−ml) + FCCCs at root node		
					# solved	% ip gap	% LB gap				# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap
25	3				3	11.30	1.21				12	16.87	4.89	8	13.53	0.36
30	3				0	21.79	1.13				0	17.58	4.65	0	16.99	0.01
15	6				0	13.99	2.49				1	12.69	4.18	0	11.96	0.99
20	6				0	15.52	0.47				0	20.30	9.11	0	14.75	0.05
		(nk−A−ml) + FCCCs + SECs			(nk−A−ml) + FCCCs			(nk−A−ml) + SECs			(nk−A−ml)			(nk−A−ml) + FCCCs at root node		
n	H	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap
25	3	3	6.23	1.34	4	5.72	1.02	6	5.17	0.72	8	4.56	0	5	5.09	0.73
30	3	1	9.75	1.03	1	9.93	0.93	1	7.53	0.48	3	6.29	0.11	1	7.33	0.5
15	6	0	6.42	0.91	0	6.06	0.88	0	4.99	0.69	0	5.59	0.02	0	5.06	0.7
20	6	0	8.23	1.2	0	8.77	1.14	0	6.21	1.03	0	6.77	0	0	6.22	1.05

Table V: FCCCs and SECs: OU policy

n	H				(k−S−ou) + FCCCs						(k−S−ou)			(k−S−ou) + FCCCs at root node		
					# solved	% ip gap	% LB gap				# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap
25	3				4	14.54	1.33				7	17.27	6.07	4	12.63	0.4
30	3				0	19.76	0.52				0	19.35	4.96	0	16.24	0.02
15	6				0	16.45	1.4				0	15.77	3.97	0	14.73	0.46
20	6				0	18.95	0.27				0	24.24	9.49	0	17.09	0
		(nk−A−ou) + FCCCs + SECs			(nk−A−ou) + FCCCs			(nk−A−ou) + SECs			(nk−A−ou)			(nk−A−ou) + FCCCs at root node		
n	H	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap	# solved	% ip gap	% LB gap
25	3	4	4.05	1.56	4	5.16	1.46	6	2.57	0.69	10	1.52	0	6	4.43	0.69
30	3	0	11.07	1.45	0	10.17	1.39	0	7.63	0.91	1	6.98	0.01	0	7.8	0.94
15	6	0	9.33	1.25	0	9.93	1.24	0	7.28	0.83	0	8.88	0.29	0	7.44	0.85
20	6	0	15.67	1.1	0	15.45	1.11	0	12	0.74	0	11.58	0.03	0	12.14	0.77

6.3 Final tests

For the final tests, we provided the exact solvers with an initial upper bound given by the heuristic solution value found by the adaptive large neighborhood search algorithm proposed in (Adulyasak et al., 2013a). Table VI reports the number of instances solved to optimality by formulations $(\mathrm{k-A-ml})$ for the ML policy and $(\mathrm{k-A-ou})$ for the OU policy. In these final tests, we removed the FCCCs as they deteriorated the performance of the algorithms. Each row of the table reports the results over five instances. We report the results for the size of the instances: $n = 20, 25, 30, 35$ for $H = 3$ and $n = 10, 15$ for $H = 6$. Note that smaller sized instances are solved by all formulations. The first four columns report the characteristics of the instances. The value h_i in the forth column indicates the inventory cost: 'L' for low and 'H' for high. In the following columns we report the results obtained for the ML and the OU policy.

Columns 5-7 refer to the ML policy. For each formulation we report the number of instances solved. Column '$(\mathrm{k-A-ml})$ (ACJ)' shows the number of instances solved in Adulyasak et al. (2013b) using the formulation $(\mathrm{k-A-ml})$.

Columns 8-11 report results for the OU policy. We include also results of formulation $(\mathrm{k-A-ou})$ because, even if it seemed to be not competitive from the preliminary test results, it turned out to be the best formulation when doing the final tests. Note that in Adulyasak et al. (2013b) formulation $(\mathrm{k-S-ou})$ was used.

Although we tested for both policies the same formulations by Adulyasak et al. (2013b), we solved to optimality a smaller number of instances. We believe that this is due to the different characteristics of the computers and to implementation details.

Table VI: Final tests: Number of instances solved to optimality

				ML			OU			
n	H	m	h_i	(k−A−ml)	(nk−A−ml)	(k−A−ml) (ACJ)	(k−A−ou)	(k−S−ou)	(nk−A−ou)	(k−S−ou)(ACJ)
20	3	2	L	5	4	5	5	5	5	5
20	3	3	L	5	2	5	5	4	3	5
25	3	2	L	5	4	5	5	5	4	5
25	3	3	L	4	0	5	5	4	0	5
30	3	3	L	3	1	5	4	0	0	5
30	3	4	L	0	1	1	1	0	0	1
35	3	3	L	1	0	5	1	0	0	2
35	3	4	L	0	0	0	0	0	0	0
Sum				23	12	(31)	26	18	12	(28)
20	3	2	H	5	4	5	5	5	5	5
20	3	3	H	5	2	5	5	5	2	5
25	3	2	H	5	4	5	5	5	4	5
25	3	3	H	5	0	5	5	4	0	5
30	3	3	H	5	1	5	2	0	0	5
30	3	4	H	1	1	2	0	0	0	1
35	3	3	H	2	0	5	0	0	0	2
35	3	4	H	0	0	0	0	0	0	0
Sum				28	12	(32)	22	19	11	(28)
10	6	2	L	5	2	5	5	5	2	5
10	6	3	L	3	2	3	0	2	1	4
15	6	2	L	5	0	5	5	4	0	5
15	6	3	L	0	0	0	0	0	0	0
Sum				13	4	(13)	10	11	3	(14)
10	6	2	H	5	2	5	5	5	2	5
10	6	3	H	3	2	3	1	1	1	4
15	6	2	H	5	0	5	5	4	0	5
15	6	3	H	**1**	0	0	0	0	0	0
Sum				14	4	(13)	11	10	3	(14)
Overall				**78**	**32**	(89)	**69**	**58**	**29**	(84)

7 Conclusions

Summarizing, we tested several vehicle index and flow formulations for the IRP under the ML and the OU policies in combination with different sets of cuts.

First and foremost, on the basis of the benchmark instances analyzed, formulations (k) that use vehicle index variables seem to be superior to the more compact flow formulations (nk). Since in all instances the number m of available vehicles does not exceed four, symmetry issues in the vehicle index formulations are not severe.

However, if it would be possible to solve extended instances with a larger vehicle fleet, e.g., $m > 5$, we expect that the relation will turn and flow formulations will become more beneficial.

With respect to the extended formulations by Solyalı and Süral (2011), our findings for the multiple-vehicle case differ from theirs for the single-vehicle case: With the ML replenishment policy, formulations $(\mathrm{k}-\mathrm{A}-\mathrm{ml})$ perform better, i.e., produce significantly more provably optimal solutions than the extended formulation $(\mathrm{k}-\mathrm{S}-\mathrm{ml})$. This coincides with the results in Adulyasak et al. (2013b), where $(\mathrm{k}-\mathrm{A}-\mathrm{ml})$ (ACJ) performs best.

For the OU policy, our study is in slight contrast to the findings of Solyalı and Süral (2011) and Adulyasak et al. (2013b) because $(\mathrm{k}-\mathrm{A}-\mathrm{ou})$ outperforms $(\mathrm{k}-\mathrm{S}-\mathrm{ou})$. We are not fully sure what causes this different outcome, but attribute the difference to implementation details such as the use of alternative subroutines and slightly different separation strategies applied in the branch-and-cut approaches.

Moreover, we tested the impact of fractional capacity-cut constraints (FCCCs) Unfortunately, the addition of cuts to the formulations has not turned out to be beneficial: They help to reduce the gap at the root node of the branch-and-cut, but overall they enlarge the models and therewith deteriorate the performance of the CPLEX MIP solver.

After all, the maximum size of instances that can be solved to optimality at the state of the art is of the order of 25-30 customers for the time horizon $H = 3$ and 10-15 customers for $H = 6$. Solving even moderately sized instances of multiple-vehicle IRP exactly remains a challenging task.

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